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LATERALLY COMPLETE REGULAR MODULES

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Abstract

Let $\mathcal{A}$ be a laterally complete commutative regular algebra and X be a laterally complete $\mathcal{A}$-module. In this paper we introduce a notion of passport $\Gamma(X)$ for X, which consist of uniquely defined partition of unity in the Boolean algebra of idempotents in $\mathcal{A}$ and the set of pairwise different cardinal numbers. It is proved that $\mathcal{A}$-modules X and Y are isomorphic if and only if $\Gamma(X) = \Gamma(Y)$.

Keywords: *Commutative regular algebra, homogeneous module, finite dimensional module*

Mathematics Subject Classification (2010): *13C05, 16D70.*

1 Introduction

J. Kaplansky [4] introduced a class of AW^*-algebras to describe C^*-algebras, which is close to von Neumann algebras by their algebraic and order structure. The class of AW^*-algebras became a subject of many researches in the operator theory (see review in [1]). One of the important results in this direction is the realization of an arbitrary AW^*-algebra M of type I as a $*$-algebra of all linear bounded operators, which act in a special Banach module over the center $Z(M)$ of the algebra M [5]. The Banach $Z(M)$-valued norm in this module is generated by the scalar product with values in the commutative AW^*-algebra $Z(M)$. Later, these modules were called Kaplansky-Hilbert modules (KHM). Detailed exposition of many useful properties of KHM is given, for example, in ([9], 7.4). One of the important properties is a representation of an arbitrary Kaplansky-Hilbert module as a direct sum of homogeneous KHM ([6], [9], 7.4.7).

Development of the noncommutative integration theory stimulated an interest to the different classes of algebras of unbounded operators, in particular, to the $*$-algebras $LS(M)$ of locally measurable operators, affiliated with von Neumann algebras or AW^*-algebras M. If M is a von Neumann algebra, then the center $Z(LS(M))$ in the algebra $LS(M)$ identifies with the algebra $L^0(\Omega, \Sigma, \mu)$ of all classes of equal almost everywhere measurable complex functions, defined on some measurable space (Ω, Σ, μ) with a complete locally finite measure μ ([11], 2.1, 2.2). If M is an AW^*-algebra, then $Z(LS(M))$ is an extended f-algebra $C_\infty(Q)$, where Q is the Stone compact coresponding to the Boolean algebra of central projectors in M [1]. The problem (like the one in the work of J. Kaplansky [5] for AW^*-algebras) on possibility of realization of $*$-algebras $LS(M)$, in the case, when M has the type I, as $*$-algebras of linear $L^0(\Omega, \Sigma, \mu)$-bounded (respectively, $C_\infty(Q)$-bounded) operators, which act in corresponding KHM over the $L^0(\Omega, \Sigma, \mu)$ or over the $C_\infty(Q)$ naturally arises. In order to solve this problem it is necessary to construct corresponding theory of KHM over the algebras $L^0(\Omega, \Sigma, \mu)$ and $C_\infty(Q)$. In a particular case of KHM over the algebras $L^0(\Omega, \Sigma, \mu)$ this problem is solved in [7], where the decomposition of KHM over $L^0(\Omega, \Sigma, \mu)$ as a direct sum of homogeneous KHM is given. Similar decomposition as a direct sum of strictly γ-homogeneous modules is given in the paper [2] for arbitrary regular laterally complete modules over the algebra $C_\infty(Q)$ (the definitions see in the Section 3 below).

The algebra $C_\infty(Q)$ is an example of a commutative unital regular algebra over the field of real numbers. In this algebra the following property of lateral completeness holds: for any set $\{a_i\}_{i\in I}$ of pairwise disjoint elements in $C_\infty(Q)$ there exists an element $a \in C_\infty(Q)$ such that $as(a_i) = a_i$ for all $i \in I$, where $s(a_i)$ is a support of the element a_i (the definitions see in the Section 2 below). This property of $C_\infty(Q)$ plays a crucial role in classification of regular laterally complete $C_\infty(Q)$-mpdules [2]. Thereby, it is natural to consider the class of laterally complete commutative unital regular algebras $\mathcal{A}$ over arbitrary fields and to obtain variants of structure theorems for modules over such algebras. Current work is devoted to solving this problem. For every faithful regular laterally complete $\mathcal{A}$-module X the concept of passport $\Gamma(X)$, which consist of the uniquely defined partition of unity in the Boolean algebra of idempotents in $\mathcal{A}$ and the set of pairwise different cardinal numbers is constructed. It is proved, that the equality of passports $\Gamma(X)$ and $\Gamma(Y)$ is necessary and sufficient condition for isomorphism of $\mathcal{A}$-modules X and Y.

2 Laterally complete commutative regular algebras

Let $\mathcal{A}$ be a commutative algebra over the field K with the unity $\mathbf{1}$ and $\nabla = \{e \in \mathcal{A} : e^2 = e\}$ be a set of all idempotents in $\mathcal{A}$. For all $e, f \in \nabla$ we write $e \leq f$ if $ef = e$. It is well known (see, for example [10, Prop. 1.6]) that this binary relation is partial order in ∇ and ∇ is a Boolean algebra with respect to this order. Moreover, we have the following equalities: $e \vee f = e + f - ef$, $e \wedge f = ef$, $Ce = \mathbf{1} - e$ with respect to the lattice operations and the complement Ce in ∇.

The commutative unital algebra $\mathcal{A}$ is called regular if the following equivalent conditions hold [12, §2, item 4]:

1. For any $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ such that $a = a^2 b$;
2. For any $a \in \mathcal{A}$ there exists $e \in \nabla$ such that $a\mathcal{A} = e\mathcal{A}$.

A regular algebra $\mathcal{A}$ is a regular semigroup with respect to the multiplication operation [3, Ch. I, §1.9]. In this case all idempotents in $\mathcal{A}$ commute pairwisely. Therefore, $\mathcal{A}$ is a commutative inverse semigroup, i.e. for any $a \in \mathcal{A}$ there exists an unique element $i(a) \in \mathcal{A}$, which is an unique solution of the system: $a^2 x = a$, $ax^2 = x$ [3, Ch. I, §1.9]. The element $i(a)$ is called an inversion of the element a. Obviously, $ai(a) \in \nabla$ for any $a \in \mathcal{A}$. In this case the map $i : \mathcal{A} \to \mathcal{A}$ is a bijection and an automorphism (by multiplication) in semigroup $\mathcal{A}$. Moreover, $i(i(a)) = a$ and $i(g) = g$ for all $a \in \mathcal{A}$, $g \in \nabla$.

Let $\mathcal{A}$ be a commutative unital regular algebra and ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. Idempotent $s(a) \in \nabla$ is called the support of an element $a \in \mathcal{A}$ if $s(a)a = a$ and $ga = a$, $g \in \nabla$ imply $s(a) \leq g$. It is clear that $s(a) = ai(a) = s(i(a))$. In particular, $s(e) = ei(e) = e$ for any $e \in \nabla$.

It is easy to show that supports of elements in a commutative regular unital algebra $\mathcal{A}$ satisfy the following properties:

Proposition 1. *Let $a, b \in \mathcal{A}$, then*
(i). $s(ab) = s(a)s(b)$, in particular, $ab = 0 \Leftrightarrow s(a)s(b) = 0$;
(ii). If $ab = 0$, then $i(a + b) = i(a) + i(b)$ and $s(a + b) = s(a) + s(b)$.

Two elements a and b in a commutative unital regular algebra $\mathcal{A}$ are called *disjoint elements*, if $ab = 0$, which equivalent to the equality $s(a)s(b) = 0$ (see Proposition 1 (i)). If the Boolean algebra ∇ of all idempotents in $\mathcal{A}$ is complete, $a \in \mathcal{A}$ and $r(a) = \sup\{e \in \nabla : ae = 0\}$, then

$$s(a)r(a) = s(a) \wedge r(a) = s(a) \wedge (\sup\{e : ae = 0\}) = \\ = \sup\{s(a) \wedge e : ae = 0\} = \sup\{s(a)e : ae = 0\} = 0.$$

Hence $s(a) \leq \mathbf{1} - r(a)$. If $q = (\mathbf{1} - r(a) - s(a))$, then $aq = as(a)q = 0$, thus $q \leq r(a)$. This yields that $q = 0$, i.e. $s(a) = \mathbf{1} - r(a)$. This implies the following

Proposition 2. *Let $\mathcal{A}$ be a commutative unital regular algebra and let ∇ be complete Boolean algebra of idempotents in $\mathcal{A}$. If $\{e_i\}_{i\in I}$ is a partition of unity in ∇, $a, b \in \mathcal{A}$ and $ae_i = be_i$ for all $i \in I$, then $a = b$.*

Proof. Since $(a - b)e_i = 0$ for any $i \in I$, then $\mathbf{1} = \sup\limits_{i\in I} e_i \leq r(a - b)$, i.e. $r(a - b) = \mathbf{1}$. Hence, $s(a - b) = 0$, i.e. $a = b$. □

Commutative unital regular algebra $\mathcal{A}$ is called *laterally complete* (*l-complete*) if the Boolean algebra of its idempotents is complete and for any set $\{a_i\}_{i\in I}$ of pairwise disjoint elements in $\mathcal{A}$ there exists an element $a \in \mathcal{A}$ such that $as(a_i) = a_i$ for all $i \in I$. The element $a \in \mathcal{A}$ such that $as(a_i) = a_i$, $i \in I$, in general, is not uniquely determined. However, by Proposition 2, it follows that the element a is unique in the case, when $\sup\limits_{i\in I} s(a_i) = \mathbf{1}$. In general case, due to the equality $as(a_i) = a_i = bs(a_i)$ for all $i \in I$ and $a, b \in \mathcal{A}$, it follows that $a \sup_{i\in I} s(a_i) = b \sup_{i\in I} s(a_i)$.

Let us give examples of l-complete and not l-complete commutative regular algebras. Let Δ be an arbitrary set and K^Δ be a Cartesian product of Δ copies of the field K, i.e. the set of all K-valued functions on Δ. The set K^Δ is a commutative unital regular algebra with respect to pointwise algebraic operations, moreover, the Boolean algebra ∇ of all idempotents in K^Δ is an isomorphic atomic Boolean algebra of all subsets in Δ. In particular ∇ is complete Boolean algebra. If $\{a_j = (\alpha_q^{(j)})_{q\in\Delta}, j \in J\}$ is a family of pairwise disjoint elements in K^Δ, then setting $\Delta_j = \{q \in \Delta : \alpha_q^{(j)} \neq 0\}$, $j \in J$ and $a = (\alpha_q)_{q\in\Delta} \in K^\Delta$, where $\alpha_q = \alpha_q^{(j)}$ for any $q \in \Delta_j$, $j \in J$, and $\alpha_q = 0$ for $q \in \Delta \setminus \bigcup\limits_{j\in J} \Delta_j$, we obtain that $as(a_j) = a_j$ for all $j \in J$. Hence, K^Δ is a l-complete algebra.

Now let $\mathcal{A}$ be an arbitrary commutative unital regular algebra over the field K and ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. An element $a \in \mathcal{A}$ is called a *step element* in $\mathcal{A}$ if it has the following form $a = \sum_{k=1}^n \lambda_k e_k$, here $\lambda_k \in K$, $e_k \in \nabla$,

$k = 1, \dots, n$. The set $K(\nabla)$ of all step elements is the smallest subalgebra in $\mathcal{A}$, which contains ∇. Any nonzero element $a = \sum_{k=1}^{n} \lambda_k e_k$ in $K(\nabla)$ can be represented as $a = \sum_{l=1}^{m} \alpha_l g_l$, here $g_l \in \nabla$, $g_l g_k = 0$ when $l \neq k$, $0 \neq \alpha_k \in K$, $l, k = 1, \dots, m$. Setting $b = \sum_{l=1}^{m} \alpha_l^{-1} g_l \in K(\nabla)$, we obtain $a^2 b = a$. Hence, $K(\nabla)$ is a regular subalgebra in $\mathcal{A}$. Since $\nabla \subset K(\nabla)$, the Boolean algebra of idempotents in $K(\nabla)$ coincides with ∇. Assume that card$(K) = \infty$ and card$(\nabla) = \infty$. We choose a countable set $K_0 = \{\lambda_n\}_{n=1}^{\infty}$ of pairwise different nonzero elements in K and a countable set $\{e_n\}_{n=1}^{\infty}$ of nonzero pairwise disjoint elements in ∇. Let us consider a set $\{\lambda_n e_n\}_{n=1}^{\infty}$ of pairwise disjoint elements in $K(\nabla)$. Assume that there exists $b = \sum_{l=1}^{m} \alpha_l g_l \in K(\nabla)$, $0 \neq \alpha_l \in K$, $g_l \in \nabla$, $g_l g_k = 0$ and $l \neq k$, $l, k = 1, \dots, m$, such that $be_n = bs(\lambda_n e_n) = \lambda_n e_n$. In this case for any positive integer n there exists integer $l(n)$, such that $\alpha_{l(n)} g_{l(n)} e_n = \lambda_n g_{l(n)} e_n \neq 0$, i.e. $\alpha_{l(n)} = \lambda_n$. This implies that the set $\{\lambda_n\}_{n=1}^{\infty}$ is finite, which is not true. Hence, the commutative unital regular algebra $K(\nabla)$ is not l-complete.

Let ∇ be complete Boolean algebra and let $Q(\nabla)$ be a Stone compact corresponding to ∇. An algebra $C_\infty(Q(\nabla))$ of all continuous functions $a : Q(\nabla) \to [-\infty, +\infty]$, taking the values $\pm\infty$ only on nowhere dense sets in $Q(\nabla)$ [9, 1.4.2], is an important example of a l-complete commutative regular algebra.

An element $e \in C_\infty(Q(\nabla))$ is an idempotent if and only if $e(t) = \chi_V(t)$, $t \in Q(\nabla)$, for some clopen set $V \subset Q(\nabla)$, where

$$\chi_V(t) = \begin{cases} 1, & t \in V; \\ 0, & t \notin V, \end{cases}$$

i.e. $\chi_V(t)$ is a characteristic function of the set V. In particular, the Boolean algebra ∇ can be identified with the Boolean algebra of all idempotents in algebra $C_\infty(Q(\nabla))$.

If $a \in C_\infty(Q(\nabla))$, then $G(a) = \{t \in Q(\nabla) : 0 < |a(t)| < +\infty\}$ is open set in the Stone compact set $Q(\nabla)$. Hence, the closure $V(a) = \overline{G(a)}$ in $Q(\nabla)$ of the set $G(a)$ is an clopen set, i.e. $\chi_{V(a)}$ is an idempotent in the algebra $C_\infty(Q(\nabla))$. We consider a continuous function $b(t)$, given on the dense open set $G(a) \cup (Q(\nabla) \setminus V(a))$ and defines by the following equation

$$b(t) = \begin{cases} \frac{1}{a(t)}, & t \in G(a), \\ 0, & t \in Q(\nabla) \setminus V(a). \end{cases}$$

This function uniquely extends to a continuous function defined on $Q(\nabla)$ with values in $[-\infty, +\infty]$ [14, Ch.5, §2] (we also denote this extension by $b(t)$). Since $ab = \chi_{V(a)}$, then $a^2 b = a$ and $s(a) = \chi_{V(a)}$. Hence, $C_\infty(Q(\nabla))$ is a commutative unital regular algebra over the field of real numbers $\mathbf{R}$. In this case, the Boolean algebra of all idempotents in $C_\infty(Q(\nabla))$ is complete.

It is known that (see, for example [9, 1.4.2]) $C_\infty(Q(\nabla))$ is an extended complete vector lattice. In particular, for any set $\{a_j\}_{j \in J}$ of pairwise disjoint positive elements in $C_\infty(Q(\nabla))$ there exists the least upper bound $a = \sup_{j \in J} a_j$ and $as(a_j) = a_j$ for all $j \in J$. It follows that the commutative regular algebra $C_\infty(Q(\nabla))$ is laterally complete.

In the case, when ∇ is a complete atomic Boolean algebra and Δ is the set of all atoms in ∇, then $C_\infty(Q(\nabla))$ is isomorphic to the algebra $\mathbf{R}^\Delta$.

The following examples of laterally complete commutative regular algebras are variants of algebras $C_\infty(Q(\nabla))$ for any topological fields, in particular, for the field $\mathbf{Q}_p$ of p-adic numbers.

Let K be an arbitrary field and t be the Hausdorff topology on K. If operations $\alpha \to (-\alpha)$, $\alpha \to \alpha^{-1}$ and operations $(\alpha, \beta) \to \alpha + \beta$, $(\alpha, \beta) \to \alpha\beta$, $\alpha, \beta \in K$, are continuous with respect to this topology, we say that (K, t) is a *topological field* (see, for example, [13, Ch.20, §165]).

Let (K, t) be a topological field, (X, τ) be any topological space and $\nabla(X)$ be a Boolean algebra of all clopen subsets in (X, τ). A map $\varphi : (X, \tau) \to (K, t)$ is called *almost continuous* if there exists a dense open set U in (X, τ) such that the restriction $\varphi|_U : U \to (K, t)$ of the map φ on the subset U is continuous in U. The set of all almost continuous maps from (X, τ) to (K, t) we denote by $AC(X, K)$.

We define pointwise algebraic operations in $AC(X, K)$ by

$$(\varphi + \psi)(t) = \varphi(t) + \psi(t);$$

$$(\alpha\varphi)(t) = \alpha\varphi(t);$$

$$(\varphi \cdot \psi)(t) = \varphi(t)\psi(t)$$

for all $\varphi, \psi \in AC(X, K)$, $\alpha \in K$, $t \in X$.

Since an intersection of two dense open sets in (X, τ) is a dense open set in (X, τ), then $\varphi + \psi$, $\varphi \cdot \psi \in AC(X, K)$ for any $\varphi, \psi \in AC(X, K)$. Obviously, $\alpha\varphi \in AC(X, K)$ for all $\varphi \in AC(X, K)$, $\alpha \in K$. It can be easily checked that $AC(X, K)$ is a commutative algebra over K with the unit element $\mathbf{1}(t) = 1_K$ for all $t \in X$, where 1_K is the unit element of K. In this case, the algebra $C(X, K)$ of all continuous maps from (X, τ) to (K, t) is a subalgebra in $AC(X, K)$.

In the algebra $AC(X, K)$ consider the following ideal

$$I_0(X, K) = \{\varphi \in AC(X, K) : \text{ interior of preimage } \varphi^{-1}(0) \text{ is dense in } (X, \tau)\}.$$

By $C_\infty(X, K)$ denote the quotient algebra $AC(X, K)/I_0(X, K)$ and by $\pi : AC(X, K) \to AC(X, K)/I_0(X, K)$ denote the corresponding canonical homomorphism.

Theorem 1. *The quotient algebra $C_\infty(X, K)$ is a commutative unital regular algebra over the field K. Moreover, if (X, τ) is a Stone compact set, then algebra $C_\infty(X, K)$ is laterally complete, and the Boolean algebra ∇ of all its idempotents is isomorphic to the Boolean algebra $\nabla(X)$.*

Proof. Since $AC(X, K)$ is a commutative unital algebra over K, then $C_\infty(X, K)$ is also a commutative unital algebra over K with unit element $\pi(\mathbf{1})$. Now we show that $C_\infty(X, K)$ is a regular algebra, i.e. for any $\varphi \in AC(X, K)$ there exists $\psi \in AC(X, K)$, such that $\pi^2(\varphi)\pi(\psi) = \pi(\varphi)$.

We fix an element $\varphi \in AC(X, K)$ and choose a dense open set $U \in \tau$, such that the restriction $\varphi|_U : U \to (K, t)$ is continuous. Since $K \setminus \{0\}$ is an open set in (K, t), then the set $V = U \cap \varphi^{-1}(K \setminus \{0\})$ is open in (X, τ). Clearly, the set $W = X \setminus \overline{V}^{\tau}$ is also open in (X, τ), in this case $V \cup W$ is a dense open set in (X, τ).

We define a map $\psi : X \to K$, as follow: $\psi(x) = (\varphi(x))^{-1}$ if $x \in V$, and $\psi(x) = 0$ if $x \in X \setminus V$. It is not hard to prove that $\psi \in AC(X, K)$ and $\varphi^2\psi - \varphi \in I_0(X, K)$, i.e. $\pi^2(\varphi)\pi(\psi) = \pi(\varphi)$. Hence, the algebra $C_\infty(X, K)$ is regular.

For any clopen set $U \in \nabla(X)$ its characteristic function χ_U belongs to $AC(X, K)$, in this case, $\pi(\chi_U)^2 = \pi(\chi_U^2) = \pi(\chi_U)$, i.e. $\pi(\chi_U)$ is an idempotent in the algebra $C_\infty(X, K)$.

Assume that (X, τ) is a Stone compact and we show that for any idempotent $e \in C_\infty(X, K)$ there exists $U \in \nabla(X)$ such that $e = \pi(\chi_U)$.

If $e \in \nabla$, then $e = \pi(\varphi)$ for some $\varphi \in AC(X, K)$ and $\pi(\varphi) = e^2 = \pi(\varphi^2)$, i.e. $(\varphi^2 - \varphi) \in I_0(X, K)$. Hence, there exists a dense open set V in X such that $\varphi^2(t) - \varphi(t) = 0$ for all $t \in V$. Denote by U a dense open set in X such that the restriction $\varphi|_U : U \to K$ is continuous. Put $U_0 = \varphi^{-1}(\{0\}) \cap (U \cap V)$, $U_1 = \varphi^{-1}(\{1_K\}) \cap (U \cap V)$. Since $U_0 \cap U_1 = \emptyset, U_0 \cup U_1 = U \cap V \in \tau$ and the sets U_0, U_1 are closed in $U \cap V$ with respect to the topology induced from (X, τ), it follows that $U_0, U_1 \in \tau$. Hence, the set $U_\varphi = \overline{U_1}$ belongs to the Boolean algebra $\nabla(X)$, besides, $U_\varphi \cap U_0 = \emptyset$.

Since $U_0 \cup U_1 = U \cap V$ is a dense open set in (X, τ) and $\varphi(t) = \chi_{U_\varphi}(t)$ for all $t \in U_0 \cup U_1$, it follows that $e = \pi(\varphi) = \pi(\chi_{U_\varphi})$. Thus, the mapping $\Phi : \nabla(X) \to \nabla$ defined by the equality $\Phi(U) = \pi(\chi_U)$, $U \in \nabla(X)$, is a surjection.

Moreover, for $U, V \in \nabla(X)$ the following equalities hold

$$\Phi(U \cap V) = \pi(\chi_{U\cap V}) = \pi(\chi_U\chi_V) = \pi(\chi_U)\pi(\chi_V) = \Phi(U)\Phi(V),$$

$$\Phi(X \setminus U) = \pi(\chi_{X\setminus U}) = \pi(\mathbf{1} - \chi_U) = \Phi(X) - \Phi(U).$$

Furthermore, the equality $\Phi(U) = \Phi(V)$ implies that the continuous mappings χ_U and χ_V coincide on a dense set in X. Therefore $\chi_U = \chi_V$, that is $U = V$.

Hence, Φ is an isomorphism from the Boolean algebra $\nabla(X)$ onto the Boolean algebra ∇ of all idempotents from $C_\infty(X, K)$, in particular, ∇ is a complete Boolean algebra.

Finally, to prove l-completeness of the algebra $C_\infty(X, K)$ we show that for any family $\{\pi(\varphi_i) : \varphi \in AC(X, K)\}_{i\in I}$ of nonzero pairwise disjoint elements in $C_\infty(X, K)$ there exists $\varphi \in AC(X, K)$ such that $\pi(\varphi)s(\pi(\varphi_i)) = \pi(\varphi_i)$ for all $i \in I$. For any $i \in I$ we choose a dense open set U_i such that the restriction $\varphi_i|_{U_i}$ is continuous and put $V_i = U_i \cap \varphi_i^{-1}(K \setminus \{0\})$, $i \in I$. It is not hard to prove that $s(\pi(\varphi_i)) = \Phi(\overline{V_i})$. In particular, $V_i \cap V_j = \emptyset$ when $i \neq j$, $i, j \in I$. Define the mapping $\varphi : X \to K$, as follows $\varphi(t) = \varphi_i(t)$ if $t \in V_i$ and $\varphi(t) = 0$ if $t \in X \setminus \left(\bigcup\limits_{i\in I} V_i\right)$. Clearly, $\varphi \in AC(X, K)$ and $\pi(\varphi)s(\pi(\varphi_i)) = \pi(\varphi\chi_{\overline{V_i}}) = \pi(\varphi_i\chi_{\overline{V_i}}) = \pi(\varphi_i)$ for all $i \in I$.

□

3 Laterally complete regular modules

Let $\mathcal{A}$ be a laterally complete commutative regular algebra and let ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. Let X be a left $\mathcal{A}$-module with algebraic operations $x+y$ and ax, $x, y \in X$, $a \in \mathcal{A}$. Since the algebra $\mathcal{A}$ is commutative, then a left $\mathcal{A}$-module X becomes a right $\mathcal{A}$-module, if we put $xa := ax$, $x \in X$, $a \in \mathcal{A}$. Hence, we can assume, that X is a bimodule over $\mathcal{A}$, where the following equality $ax = xa$ holds for any $x \in X$, $a \in \mathcal{A}$. Next, an $\mathcal{A}$-bimodule X we shall call an $\mathcal{A}$-module.

An $\mathcal{A}$-module X is called faithful, if for any nonzero $e \in \nabla$ there exists $x \in X$ such that $ex \neq 0$. Clearly, for a faithful $\mathcal{A}$-module X the set $X_e := eX$ is a faithful $\mathcal{A}_e$-module for any $0 \neq e \in \nabla$, where $\mathcal{A}_e := e\mathcal{A}$.

An $\mathcal{A}$-module X is said to be a regular module, if for any $x \in \mathcal{A}$ the condition $ex = 0$, for all $e \in L \subset \nabla$, implies $(\sup L)x = 0$. In this case, for $x \in X$ the idempotent $s(x) = \mathbf{1} - \sup\{e \in \nabla : ex = 0\}$ is called the support of an element x. In case, when $X = \mathcal{A}$, the notions of support of an element in an $\mathcal{A}$-module X and of support of an element in $\mathcal{A}$ coincide. If X is a regular $\mathcal{A}$-module, then X_e is also a regular $\mathcal{A}_e$-module for any nonzero $e \in \nabla$.

We need the following properties of supports of elements in a regular $\mathcal{A}$-module X.

Proposition 3. *Let X be a regular $\mathcal{A}$-module, $x, y \in X$, $a \in \mathcal{A}$. Then*
(i). $s(x)x = x$;
(ii). if $e \in \nabla$ and $ex = x$, then $e \geq s(x)$;
(iii). $s(ax) = s(a)s(x)$.

Proof. (i). If $r(x) = \sup\{e \in \nabla : ex = 0\}$, then $s(x) = \mathbf{1} - r(x)$ and $r(x)x = 0$. Hence, $x = (s(x) + r(x))x = s(x)x$.

(ii). As $ex = x$, then $(\mathbf{1} - e)x = 0$, and therefore $\mathbf{1} - e \leq r(x)$. This implies $e \geq \mathbf{1} - r(x) = s(x)$.

(iii). Since $(s(a)s(x)) \cdot (ax) = (s(a)a) \cdot (s(x)x) = ax$, then by (ii) we have $s(ax) \leq s(a)s(x)$. If $g = s(a)s(x) - s(ax) \neq 0$, then $ga \neq 0$, $g \leq s(a)$ and $gs(ax) = 0$. Hence $gax = 0$ and $0 = i(ga)(gax) = (i(g)i(a)ga)x = (gi(a)a)x = gs(a)x = gx \neq 0$. This contradiction implies $g = 0$, i.e. $s(ax) = s(a)s(x)$. □

We say that a regular $\mathcal{A}$-module X is laterally complete (l-complete), if for any set $\{x_i\}_{i\in I} \subset X$ and for any partition $\{e_i\}_{i\in I}$ of unity of the Boolean algebra ∇ there exists $x \in X$ such that $e_i x = e_i x_i$ for all $i \in I$. In this case, the element x is called mixing of the set $\{x_i\}_{i\in I}$ with respect to the partition of unity $\{e_i\}_{i\in I}$ and denote by $\underset{i\in I}{\operatorname{mix}}(e_i x_i)$. Mixing $\underset{i\in I}{\operatorname{mix}}(e_i x_i)$ is defined uniquely, whereas the equalities $e_i x = e_i x_i = e_i y$, $x, y \in X$, $i \in I$, implies $e_i(x - y) = 0$ for all $i \in I$, and, by regularity of the $\mathcal{A}$-module X, we obtain $x = y$.

Let $\{x_i\}_{i\in I} \subset E \subset X$ and let $\{e_i\}_{i\in I}$ be a partition of unity in ∇. The set of all mixings $\underset{i\in I}{\operatorname{mix}}(e_i x_i)$ is called a cyclic hull of the set E in X and denotes by $\operatorname{mix}(E)$. Obviously, the inclusion $E \subset \operatorname{mix}(E)$ is always true. If $E = \operatorname{mix}(E)$, then E is called a cyclic set in X (compare with [8], 1.1.2).

Thus, a regular $\mathcal{A}$-module X is a l-complete $\mathcal{A}$-module if and only if X is a cyclic set. In particular, in any l-complete $\mathcal{A}$-module X its submodule X_e is also a l-complete $\mathcal{A}_e$-module for any nonzero idempotent e in $\mathcal{A}$.

We need the following properties of cyclic hulls of sets.

Proposition 4. *Let X be a l-complete $\mathcal{A}$-module and let E be a nonempty subset in X, $a \in \mathcal{A}$. Then*

(i). $\mathrm{mix}(\mathrm{mix}(E)) = \mathrm{mix}(E)$*;*

(ii). $\mathrm{mix}(aE) = a\mathrm{mix}(E)$*;*

(iii). If Y is an $\mathcal{A}$-submodule in X, then $\mathrm{mix}(Y)$ *is a l-complete $\mathcal{A}$-submodule in X;*

(iv). If U is an isomorphism from $\mathcal{A}$-module X onto $\mathcal{A}$-module Z, then Z is a l-complete $\mathcal{A}$-module and $\mathrm{mix}(U(E)) = U(\mathrm{mix}(E))$.

Proof. (i). It is sufficient to show that $\mathrm{mix}(\mathrm{mix}(E)) \subset \mathrm{mix}(E)$. If $x \in \mathrm{mix}(\mathrm{mix}(E))$, then $x = \underset{i\in I}{\mathrm{mix}}(e_i x_i)$, where $x_i \in \mathrm{mix}(E)$, $i \in I$. Since $x_i \in \mathrm{mix}(E)$, then $x_i = \underset{j\in J(i)}{\mathrm{mix}}(e_j^{(i)} x_j^{(i)})$, where $x_j^{(i)} \in E$, $j \in J(i)$ and $\{e_j^{(i)}\}_{j\in J(i)}$ is a partition of unity in the Boolean algebra ∇ for all $i \in I$. Fix $i \in I$ and put $g_j^{(i)} := e_i e_j^{(i)}$. It is clear that $\{g_j^{(i)}\}_{j\in J(i)}$ is a partition of the idempotent e_i. Hence, $\{g_j^{(i)}\}_{j\in J(i), i\in I}$ is a partition of unity $\mathbf{1}$. Besides, $g_j^{(i)} x = g_j^{(i)} e_i x = g_j^{(i)} e_i x_i = e_i e_j^{(i)} x_i = e_i e_j^{(i)} x_j^{(i)} = g_j^{(i)} x_j^{(i)}$. This yields that $x = \underset{j\in J(i), i\in I}{\mathrm{mix}}(g_j^{(i)} x_j^{(i)}) \in \mathrm{mix}(E)$.

(ii). If $x \in \mathrm{mix}(aE)$, then $x = \underset{i\in I}{\mathrm{mix}}(e_i a y_i)$, where $y_i \in E, i \in I$. Since X is a l-complete $\mathcal{A}$-module, then there exists $y = \underset{i\in I}{\mathrm{mix}}(e_i y_i) \in \mathrm{mix}(E)$ and $e_i x = a e_i y_i = e_i(ay)$ for all $i \in I$. Hence, $e_i(x - ay) = 0$, and regularity of the $\mathcal{A}$-module X implies the equality $x = ay$. Thus, $\mathrm{mix}(aE) \subset a\mathrm{mix}(E)$.

Conversely, if $x \in a\mathrm{mix}(E)$, then $x = az$, where $z = \underset{i\in I}{\mathrm{mix}}(e_i z_i), z_i \in E, i \in I$. Since $az_i \in aE$ and $e_i x = e_i(az) = e_i a e_i z = e_i(az_i)$ for all $i \in I$, we have that $x = \underset{i\in I}{\mathrm{mix}}(e_i(az_i)) \in \mathrm{mix}(aE)$. Hence, $a\mathrm{mix}(E) \subset \mathrm{mix}(aE)$.

(iii). Let $x, y \in \mathrm{mix}(Y)$, $x = \underset{i\in I}{\mathrm{mix}}(e_i x_i)$, $y = \underset{j\in J}{\mathrm{mix}}(g_j y_j)$, where $x_i, y_j \in Y$, $i \in I$, $j \in J$, $\{e_i\}_{i\in I}$, $\{g_j\}_{j\in J}$ are partitions of unity in ∇. Clearly, that $p_{ij} = e_i g_j$, $i \in I$, $j \in J$, is also a partition of unity in ∇ and $p_{ij}(x+y) = p_{ij}(x_i + y_j)$, where $x_i + y_j \in Y$ for all $i \in I$, $j \in J$. This means that $(x + y) \in \mathrm{mix}(Y)$.

Since $aY \subset Y$, then by (ii) we have that $ax \in a\mathrm{mix}(Y) = \mathrm{mix}(aY) \subset \mathrm{mix}(Y)$. Hence, $\mathrm{mix}(Y)$ is an $\mathcal{A}$-submodule in X, and by regularity of the $\mathcal{A}$-module X, it is a regular $\mathcal{A}$-module. The equality $\mathrm{mix}(Y) = \mathrm{mix}(\mathrm{mix}Y)$ (see (i)) implies that $\mathrm{mix}Y$ is a l-complete $\mathcal{A}$-module.

(iv). If $U(x) = y \in Z$, $x \in X$, $\emptyset \neq L \subset \nabla$ and $ey = 0$ for all $e \in L$, then $U(ex) = eU(x) = ey = 0$. Since U is a bijection, then $ex = 0$ for any $e \in L$. By regularity of the $\mathcal{A}$-module X, we have that $(\sup L)x = 0$, and, therefore, $(\sup L)y = U((\sup L)x) = 0$. Hence, Z is a regular $\mathcal{A}$-module. In the same way we show that Z is a l-complete $\mathcal{A}$-module and the equality $\mathrm{mix}(U(E)) = U(\mathrm{mix}(E))$ holds.

□

Let ∇ be an arbitrary complete Boolean algebra. For any nonzero element $e \in \nabla$ we put $\nabla_e = \{q \in \nabla : q \leq e\}$. The set ∇_e is a Boolean algebra with the unity e with respect to partial order, induced from ∇.

We say that a set B in ∇ is a minorant subset for nonempty set $E \subset \nabla$, if for any nonzero $e \in E$ there exists nonzero $q \in B$ such that $q \leq e$. We need the following property of complete Boolean algebras.

Theorem 2. ([9], 1.1.6) *If ∇ is a complete Boolean algebra, e is a nonzero element in ∇ and B is a minorant subset for ∇_e, then there exists a disjoint subset $L \subset B$ such that* $\sup L = e$.

We say that a Boolean algebra ∇ has *a countable type* or is *σ-finite*, if any nonfinite family of nonzero pairwise disjoint elements in ∇ is a countable set. A complete Boolean algebra ∇ is called *multi-σ-finite*, if for any nonzero element $g \in \nabla$ there exists $0 \neq e \in \nabla$ such that $e \leq g$ and the Boolean algebra ∇_e has a countable type. By theorem 2, a multi-σ-finite Boolean algebra ∇ always has a partition $\{e_i\}_{i\in I}$ of unity $\mathbf{1}$ such that the Boolean algebra ∇_{e_i} has a countable type for all $i \in I$.

By theorem 2 we set the following useful properties of l-complete $\mathcal{A}$-modules.

Proposition 5. *Let X be an arbitrary l-complete $\mathcal{A}$-module and ∇ be a complete Boolean algebra of all idempotents in $\mathcal{A}$. Then*

(i). If X is a faithful $\mathcal{A}$-module, then there exists an element $x \in X$ such that $s(x) = \mathbf{1}$;

(ii). If Y is a l-complete $\mathcal{A}$-submodule in a regular $\mathcal{A}$-module X and for any nonzero $e \in \nabla$ there exists a nonzero $g_e \in \nabla$ such that $g_e \leq e$ and $g_eY = g_eX$, then $Y = X$.

Proof is in the same way as the proof of Proposition 2.4 in [2].

We need a representation of a faithful l-complete $\mathcal{A}$-module X as the Cartesian product of a faithful l-complete $\mathcal{A}_{e_i}$-modules family, where $\{e_i\}_{i\in I}$ is a partition of unity in the Boolean algebra ∇ of all idempotents in $\mathcal{A}$. In the Cartesian product

$$\prod_{i\in I} e_iX = \{\{y_i\}_{i\in I} : y_i \in e_iX\}$$

of $\mathcal{A}$-submodules e_iX we consider coordinate-wise algebraic operations. It is clear that $\prod_{i\in I} e_iX$ is a faithful l-complete $\mathcal{A}$-module. We define a map $U : X \to \prod_{i\in I} e_iX$ given by $U(x) = \{e_ix\}_{i\in I}$. Obviously, U is a linear mapping from X onto $\prod_{i\in I} e_iX$. If $U(x) = U(y)$, then $e_ix = e_iy$ for all $i \in I$, and by regularity of the $\mathcal{A}$-module X, it follows that $x = y$.

If $z = \{x_i\}_{i\in I} \in \prod_{i\in I} e_iX$, where $x_i \in e_iX \subset X$, $i \in I$, then l-completeness of the $\mathcal{A}$-module X implies that there exists an element $x \in X$ such that $e_ix = e_ix_i = x_i$ for all $i \in I$. Hence, $U(x) = z$, i.e. U is a surjection.

Thus, the following proposition holds.

Proposition 6. *If X is a faithful l-complete $\mathcal{A}$-module, $\{e_i\}_{i\in I}$ is a partition of unity of the Boolean algebra ∇ of all idempotents in $\mathcal{A}$, then $\prod\limits_{i\in I} e_iX$ is also a faithful l-complete $\mathcal{A}$-module and U is an isomorphism from X onto $\prod\limits_{i\in I} e_iX$.*

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