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Jasurbek Karimov

CYCLICALLY COMPACT OPERATORS IN BANACH MODULES OVER $L^0(B)$

Jasurbek Karimov
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: karimovja@mail.ru, ja.karimov@nuu.uz

Abstract

In this paper the properties of linear cyclically compact operators in Banach modules over space $L^0(B)$ are given.

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1 Introduction

In [9] were considered the cyclically compact linear operators acting in the Banach–Kantorovich spaces $(X, \|\,.\,\|_X)$, whose norm values are in the order-complete vector lattice E. An important class of such lattices E is formed by Kantorovich–Pinsker spaces ([9]). These spaces admit an R-topology, with respect to which E becomes a separable topological vector space ([10]). The presence of the R-topology allows one to naturally define a separable vector topology $\tau(X)$ on the the Banach–Kantorovich space X, turning X into a topological E-module. This paper describes the properties of linear cyclically compact operators acting in topological E-modules $(X, \tau(X))$ for extended Kantorovich–Pinsker spaces E.

2 Cyclically compact sets

Let B be a multinormed Boolean algebra ([9]), $L^0(B)$ be an extended Kantorovich–Pinsker space associated with B, X be a left unitary $L^0(B)$-module ([4, 1, 5, 3, 2, 6, 7]), $\|\,.\,\|_X : X \to L^0(B)$ is a norm on X endowing X with the structure of a Banach $L^0(B)$-module. We denote by $t(B)$ the R-topology in $L^0(B)$ ([10]) and for each neighborhood of zero of U in $(L^0(B), \tau(B))$ we put $W(U) = \{x \in X : \|x\|_X \in U\}$. According to ([11]), in X there is a topology $\tau(X)$, with respect to which $(X, \tau(X))$ is a separable topological vector space, and, moreover, the system of sets $\{x + W(U)\}$ forms a basis for neighborhoods of the element $x \in X$ (in this case the topology $\tau(X)$ is said to be generated by the norm $\|\,.\,\|_X$ and R-topology $t(B)$). The convergence of the network $x_\alpha \xrightarrow{\tau(X)} x$, $x_\alpha, x \in X$ means that $\|x_\alpha - x\|_X \xrightarrow{t(B)} 0$.

Let $\{e_i\}_{i\in I}$ be an arbitrary partition of the unity $\mathbf{1}$ in the Boolean algebra B of all idempotents from $L^0(B)$, $\{x_i\}_{i\in I} \subset X$, where I is a certain set of indices. An element $x \in X$ is called a mixing of the family $\{x_i\}$ with respect to $\{e_i\}$ if $e_i x = e_i x_i$ for all $i \in I$ ([8]). Since $(X, \|\,.\,\|_X)$ is a Banach $L^0(B)$-module, then mixing always exists and it is unique ([9]). Mixing x is denoted by $\underset{i\in I}{\text{mix}}\,(e_i x_i)$.

The set of all mixings $\underset{i\in I}{\text{mix}}\,(e_i x_i)$, where $\{x_i\}_{i\in I} \subset F \subset X$, $\{e_i\}_{i\in I}$ is a partition of unity in B, is called the cyclic hull of a subset F from X and is denoted by $\text{mix}(F)$. If $F = \text{mix}(F)$, then F is said to be a cyclic subset of X ([8]). It is clear that $X = \text{mix}(X)$.

Let B be a multinormed Boolean algebra and let $P(\mathbb{N})$ be a set of all countable partitions of unity in B, numbered by positive integers $n \in \mathbb{N}$, i.e.

$$P(\mathbb{N}) = \left\{ a : \mathbb{N} \to B \,|\, a(n) \wedge a(m) = 0, n \neq m, \sup_{n\in\mathbb{N}} a(n) = \mathbf{1} \right\}.$$

We introduce a partial order into $P(\mathbb{N})$, setting $a \leq b \Leftrightarrow$ (for any $n, m \in \mathbb{N}$ from $a(n) \wedge b(m) \neq 0$ it follows, that $n \leq m$, where $a, b \in P(\mathbb{N})$). In [8] it is shown that the introduced relation $a \leq b$ is a partial order relation on $P(\mathbb{N})$ and the partially ordered set $(P(\mathbb{N}), \leq)$ is the direction, i.e. for any $a, b \in P(\mathbb{N})$ there is $d \in P(\mathbb{N})$ such that $a \leq d$, $b \leq d$.

Let $\{x_n\}_{n\in N}$ be a sequence of X. For each $a \in P(\mathbb{N})$ we put $x_a = \underset{n\in\mathbb{N}}{\text{mix}}\,(a(n)\,x_n)$. Any subsequence of the net $\{x_a\}_{a\in P(A)}$ is called a cyclic subsequence of the original sequence $\{x_n\}_{n\in\mathbb{N}}$.

A subset $K \subset X$ is said to be cyclically compact if $K = \text{mix}(K)$ and every sequence in K has a cyclic subsequence $\tau(X)$-converging to some element in K [7]. A subset $K \subset X$ is said to be relatively cyclically compact if K is contained in some cyclically compact set of X [7].

Proposition 1. *Let X be a Banach $L^0(B)$-module, $f \in L^0$, $K \subset X$. If K is cyclically compact (respectively, relatively cyclically compact), then $fK = \{fx : x \in K\}$ is also cyclically compact (respectively, relatively cyclically compact).*

Proof. First let K be a cyclically compact. For any partition $\{e_i\}_{i\in I}$ of the unity and the set $\{fx_i\}_{i\in I} \subset fK$, $x_i \in K$, we have that for

$$\underset{i\in I}{\text{mix}}\,(e_i f\, x_i) = f\underset{i\in I}{\text{mix}}\,(e_i\, x_i) \in fK.$$

Therefore, $\text{mix}(fK) = fK$.

Consider an arbitrary sequence $y_n = fz_n \in fK$, $z_n \in K$. Since K is cyclically compact, there exists a cyclic subsequence $\{z_{a_k}\}_{k\in N}$ and an element $x \in K$ such that $\|z_{a_k} - x\|_X \xrightarrow{t(B)} 0$. It is clear that $y_{a_k} = fz_{a_k}$ is a cyclic subsequence for the sequence $\{y_n\}_{n\in N}$, while $\|y_{a_k} - fx\|_X \xrightarrow{t(B)} 0$ and $fx \in K$. This means that fK is a cyclically compact set.

The relative cyclic compactness of the set fK is established similarly in the case when K is relatively cyclically compact. □

We need the following useful criterion for relative cyclic compactness from [9].

Theorem 1. *Let K be an arbitrary cyclic set from the Banach $L^0(B)$-module $(X, \|\,.\,\|_X)$. Then K is relatively cyclically compact if and only if for any $\varepsilon > 0$ there exist a countable partition $\{e_n\}_{n\in\mathbb{N}}$ of the unity in the Boolean algebra B and the sequence $\{E_n\}_{n\in\mathbb{N}}$*

of finite subsets of $E_n = \{x_1^{(n)}, \dots, x_{k(n)}^{(n)}\} \subset K$ *such that* $e_n(\text{mix}(E_n))$ *serves as a* ε-*net for* e_nK *for all* $n \in \mathbb{N}$, *i.e. for each* $x \in e_nK$ *there is a partition* $\{q_1^{(n)}, \dots, q_{k(n)}^{(n)}\}$ *of the unity in* B, *for which*

$$\left\| x - \sum_{i=1}^{k(n)} e_n q_i^{(n)} x_i^{(n)} \right\|_X \leq \varepsilon \cdot \mathbf{1}.$$

A set $F \subset X$ is called L^0-bounded if $\|x\|_X \leq f$ for all $x \in F$ and some $0 \leq f \in L^0(B)$. The following proposition follows from Theorem 1.

Proposition 2. *Every relatively cyclically compact set is* L^0-*bounded.*

3 Cyclically compact operators

Let $(X, \|\,.\,\|_X)$ and $(Y, \|\,.\,\|_Y)$ be Banach $L^0(B)$-modules. A linear operator $T : X \to Y$ is called L^0-bounded if there exists $0 \leq f \in L^0(B)$ such that $\|Tx\|_Y \leq f\|x\|_X$ for all $x \in X$. We denote by $B(X, Y)$ the left unitary $L^0(B)$-module of all L^0-bounded linear mappings from X to Y. With respect to the norm $\|T\|_{B(X,Y)} = \sup\limits_{\|x\|_X \leq 1} \|Tx\|_Y$ the module $B(X, Y)$ becomes a Banach $L^0(B)$-module. Let $\tau(B(X, Y))$ denote the topology in $B(X, Y)$ generated by the norm $\|\,.\,\|_{B(X,Y)}$ and R-topology $t(B)$.

Each L^0-bounded operator T has the following property of L^0-linearity ([9]): $T(fx + gy) = fT(x) + gT(y)$ for any $f, g \in L^0$, $x, y \in X$. The following propositions holds

Proposition 3. *A* L^0-*linear operator* T *from* X *to* Y *is* L^0-*bounded if and only if the set* $T(\{x \in X : \|x\|_X \leq 1\})$ *is* L^0-*bounded in* Y.

A mapping $V : X \to Y$ is said to preserve mixing if $V(\operatorname{mix}\limits_{i \in I}(e_i x_i)) = \operatorname{mix}\limits_{i \in I}(e_i T(x_i))$ for any partition $\{e_i\}_{i \in I}$ of the unity and an arbitrary set $\{x_i\}_{i \in I} \subset X$. The following proposition gives examples of mixing-preserving mappings.

Proposition 4. *(i) If* $T \in B(X, Y)$ *then* T *preserves mixing;*

(ii) A map $\|\,.\,\|_X : X \to L^0(B)$ *preserves mixing.*

Proof. (i). If $T \in B(X, Y)$, then T is L^0-linear, and therefore for $x = \operatorname{mix}\limits_{i \in I}(e_i x_i)$ we have that $e_i x = e_i x_i$ and $e_i T(x) = T(e_i x) = T(e_i x_i) = e_i T(x_i)$. Hence,

$$\operatorname{mix}_{i \in I}(e_i T(x_i)) = T(x) = T(\operatorname{mix}_{i \in I}(e_i x_i)),$$

i.e. T preserves mixing.

The statement (ii) is proved in a similar way. □

Let $U(0, \mathbf{1}) = \{x \in X : \|x\|_X \leq 1\}$. The following corollary follows from Proposition 4

Corollary 1. *If $T \in B(X,Y)$, then $T(U(0,\mathbf{1}))$ is a cyclic set.*

Proof. Let $\{e_i\}_{i\in I} \in P(A)$, $\{y_i\}_{i\in I} \subset T(U(0,\mathbf{1}))$, $y_i = T(x_i)$, where $x_i \in U(0,\mathbf{1})$, $i \in I$. For $y = \operatorname*{mix}_{i\in I}(e_i y_i)$ and $x = \operatorname*{mix}_{i\in I}(e_i x_i)$, by Proposition 4 (i), we have that $T(x) = y$, and $\|x\|_X = \operatorname*{mix}_{i\in I}(e_i\|x_i\|_X) \le 1$, i.e. $y \in T(U(0,\mathbf{1}))$. Therefore, $\operatorname{mix}(T(U(0,\mathbf{1}))) = T(U(0,\mathbf{1}))$. □

A L^0-linear operator $T : X \to Y$ is called cyclically compact if for any L^0-bounded set $F \subset X$ its image $T(F)$ is relatively cyclically compact in Y.

Let $K(X,Y)$ denote the set of all L^0-linear cyclically compact operators from X to Y. From Propositions 2 and 3 it follows that $K(X,Y) \subset B(X,Y)$.

Proposition 5. *If $T : X \to Y$ is a L^0-linear map and the image of $T(\{x \in X : \|x\|_X \le \mathbf{1}\})$ is relatively cyclically compact, then $T \in K(X,Y)$.*

Proof. According to Propositions 2 and 3, we have that $T \in B(X,Y)$. Let F be a L^0-bounded set in X, i.e., there exists a $0 \le g \in L^0(B)$, that $\|x\|_X \le g$ for all $x \in F$. Let $G = \{(\mathbf{1}+g)^{-1}x \,|\, x \in F\}$. Since $\|(\mathbf{1}+g)^{-1}x\|_X = (\mathbf{1}+g)^{-1}\|x\|_X \le 1$ for all $x \in F$, then $G \subset U(0,\mathbf{1})$, and therefore $T(G) \subset T(U(0,\mathbf{1}))$. Since $T(U(0,\mathbf{1}))$ is relatively cyclically compact, the set $T(G) = (\mathbf{1}+g)^{-1}T(F)$ is also relatively cyclically compact. According to Proposition 1, we obtain that the set $T(F) = (\mathbf{1}+g)T(G)$ is also relatively cyclically compact, i.e. $T \in K(X,Y)$. □

Theorem 2. *$K(X,Y)$ is a $\tau(B(X,Y))$-closed $L^0(B)$-submodule of $B(X,Y)$.*

Proof. If $T, S \in K(X,Y)$, then the sets $K_1 = T(U(0,\mathbf{1}))$, $K_2 = S(U(0,\mathbf{1}))$ are relatively cyclically compact in Y, while K_1 and K_2 are cyclic (see Corollary 1). In addition, for any $e \in B$ we have that $eK_1 = T(eU(0,\mathbf{1})) \subset T(U(0,\mathbf{1}))$, i.e. $eK_1 \subset K_1$. Similarly, $eK_2 \subset K_2$. According to Theorem 1, for any $\varepsilon > 0$ there are countable partitions $\{e_n\}_{n\in\mathbb{N}}$ and $\{e'_n\}_{n\in\mathbb{N}}$ of the unity in B and sequences of finite sets

$$E_n = \{x_1^{(n)}, \dots, x_{k(n)}^{(n)}\} \subset K_1$$

and

$$F_n = \{y_1^{(n)}, \dots, y_{s(n)}^{(n)}\} \subset K_2$$

such that $e_n(\operatorname{mix}(E_n))$ and $e'_n(\operatorname{mix}(F_n))$ are $\varepsilon/2$ -nets for e_nK_1 and for e'_nK_2, respectively, for any $n \in \mathbb{N}$.

Let

$$q_{n,m} = e_n \cdot e'_m,\ D_{n,m} = E_n + F_m = \{x_i^{(n)} + y_j^{(m)} : i = \overline{1,k(n)}, j = \overline{1,s(m)}\}.$$

It is clear that $D_{n,m}$ is a finite subset of $K_1 + K_2 = (T+S)(U(0,\mathbf{1}))$, $q_{n,m} \cdot q_{n',m'} = 0$, if $(n,m) \ne (n',m')$, $\sup_{n,m\in\mathbb{N}} q_{n,m} = 1$, i.e. $\{q_{n,m}\}$ is a countable partition of the unity in B. It is easy to verify that $q_{n,m}(\operatorname{mix}(D_{n,m}))$ is a ε-net for the set $q_{n,m}(K_1+K_2)$. Moreover, $K_1+K_2 = (T+S)(U(0,\mathbf{1}))$ is a cyclic set (see Corollary 1). Therefore, by Theorem 1, the set $(T+S)(U(0,\mathbf{1}))$ is relatively cyclically compact, which, according to Proposition 5, implies the cyclic compactness of the operator $T+S$, i.e. $T+S \in K(X,Y)$.

Since $T(U(0,\mathbf{1}))$ is relatively cyclically compact, then, according to Proposition 1, the set $(fT)(U(0,\mathbf{1})) = fT(U(0,\mathbf{1}))$ is also relatively cyclically compact, and therefore $fT \in K(X,Y)$ (see Proposition 5) for any $f \in L^0(B)$.

Thus $K(X,Y)$ is a $L^0(B)$-submodule of $B(X,Y)$.

Now we show that $K(X,Y)$ is a $\tau(B(X,Y))$-closed $L^0(B)$-submodule in $B(X,Y)$. Let $T_\alpha \in K(X,Y)$, $T \in B(X,Y)$ and $\|T_\alpha - T\|_{B(X,Y)} \xrightarrow{t(B)} 0$. Since B is a multi-normed Boolean algebra, then there is a partition $\{e_i\}_{i\in I}$ of the unity in B such that the Boolean algebra e_iB has a countable type for each $i \in I$. It is clear that $e_iT_\alpha \in K(e_iX, e_iY)$, $e_iT \in B(e_iX, e_iY)$, while $\|e_iT_\alpha - e_iT\|_{B(e_iX,e_iY)} \xrightarrow{t(B)} 0$ for all $i \in I$.

We show that $e_iT \in K(e_iX, e_iY)$, $i \in I$. Since e_iB is of countable type, there is a sequence of indices $\alpha_1 \le \ldots \le \alpha_n \le \ldots$ such that

$$\|e_iT_{\alpha_n} - e_iT\|_{B(e_iX,e_iY)} \xrightarrow{(o)} 0.$$

From ([12]) it follows that there exist such $0 \le u \in L^0(e_iB)$ and numbers $\varepsilon_n \downarrow 0$, which means

$$\|e_iT_{\alpha_n} - e_iT\|_{B(e_iX,e_iY)} \le \varepsilon_n u.$$

Let $S_n = (e_i+u)^{-1}e_iT_{\alpha_n}$, $S = (e_i+u)^{-1}e_iT$. We have that $S_n \in K(e_iX, e_iY)$, $S \in B(e_iX, e_iY)$ and

$$\|S_n - S\|_{B(e_iX,e_iY)} = (e_i+u)^{-1}\|T_{\alpha_n} - T\|_{B(e_iX,e_iY)} \le \varepsilon_n(e_i+u)^{-1} \le \varepsilon_n \cdot e_i$$

for all $n \in \mathbb{N}$.

We fix $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0} < \frac{\varepsilon}{2}$, i.e.

$$\|S_{n_0} - S\|_{B(e_iX,e_iY)} \le \frac{\varepsilon}{2} \cdot e_i.$$

Since $S_{n_0} \in K(e_iX, e_iY)$, then there is a countable partition $\{p_n\}_{n\in N}$ of idempotent e_i and sequence of finite sets $E_n = \{y_1^{(n)}, \ldots, y_{k(n)}^{(n)}\} \subset S_{n_0}(U(0,e_i))$ such that $p_n(\mathrm{mix}(E_n))$ is a $\varepsilon/2$-net for $p_nS(U(0,e_i))$. Let $y \in p_nS(U(0,e_i))$, i.e. $y = p_nS(x)$, where $x \in U(0,e_i)$. We put $z = p_nS_{n_0}(x)$. Then $z \in p_nS_{n_0}(U(0,e_i))$ and therefore there is a partition $\{q_1^{(n)}, \ldots, q_{k(n)}^{(n)}\}$ of the unity in B such that

$$\left\| z - \sum_{i=1}^{k(n)} p_n q_i^{(n)} y_i^{(n)} \right\|_X \le \frac{\varepsilon}{2} \cdot e_i.$$

Hence,

$$\left\| y - \sum_{i=1}^{k(n)} p_n q_i^{(n)} y_i^{(n)} \right\|_X = \left\| p_nS(x) - p_nS_{n_0}(x) + z - \sum_{i=1}^{k(n)} p_n q_i^{(n)} y_i^{(n)} \right\|_X \le$$

$$\le p_n\|S(x) - S_{n_0}\|_{B(X,Y)} \cdot \|x\|_X + \left\| z - \sum_{i=1}^{k(n)} p_n q_i^{(n)} y_i^{(n)} \right\|_X \le \varepsilon \cdot e_i.$$

According to Theorem 1, we have that $S(U(0,e_i))$ is relatively cyclically compact, and therefore $S \in K(e_iX, e_iY)$ (see Proposition 5). Therefore, $e_iT = (e_i + u)S \in K(e_iX, e_iY)$ for all $i \in I$, and therefore $T \in K(X,Y)$. □

We put $K(X) = K(X,X)$, $B(X) = B(X,X)$.

Theorem 3. *If $T \in K(X)$, $S \in B(X)$, then $TS, ST \in K(X)$.*

Proof. Since $S \in B(X)$, the set $S(U(0,\mathbf{1}))$ is L^0-bounded (see Proposition 3). Since $T \in K(X)$, then $(TS)(U(0,\mathbf{1})) = T(S(U(0,\mathbf{1})))$ is a relatively cyclically compact set in X. By Proposition 5, we have that $TS \in K(X)$.

Now we show that $ST \in K(X)$. Let $S_1 = (\mathbf{1} + \|S\|)^{-1}S$. It is clear that $S_1 \in B(X)$ and $\|S_1\| \leq \mathbf{1}$. Consider the set $M = (S_1T)(U(0,\mathbf{1}))$. According to Corollary 1, the set M is cyclic. Since $T \in K(X)$, then, by Theorem 1 and Corollary 1, there is a countable partition $\{e_n\}_{n\in\mathbb{N}}$ of the unity in B and a sequence of finite subsets $E_n = \{y_1^{(n)}, \dots, y_{k(n)}^{(n)}\} \subset T(U(0,\mathbf{1}))$ such that $e_n(\underset{n\in\mathbb{N}}{\mathrm{mix}}\,(E_n))$ is a ε-net for $e_nT(U(0,\mathbf{1}))$. Let us show that $e_n(\underset{n\in\mathbb{N}}{\mathrm{mix}}\, S_1(E_n))$ is a ε-net for M. Let $y \in e_nM$, i.e. $y = e_nS_1(T(x))$, where $\|x\|_X \leq \mathbf{1}$. Let $z = e_nT(x)$. Then $z \in e_nT(U(0,\mathbf{1}))$, and therefore there is a partition $\{q_1^{(n)}, \dots, q_{k(n)}^{(n)}\}$ of the unity in B, such that

$$\left\| z - \sum_{i=1}^{k(n)} e_n q_i^{(n)}(y_i^{(n)}) \right\|_X \leq \varepsilon \cdot \mathbf{1}.$$

Since $y = e_nS_1(T(x)) = S_1(e_nT(x)) = S_1(z)$ and $\|S_1\| \leq \mathbf{1}$, then

$$\left\| y - \sum_{i=1}^{k(n)} e_n q_i^{(n)} S_1(y_i^{(n)}) \right\|_X = \left\| S_1\left(z - \sum_{i=1}^{k(n)} e_n q_i^{(n)} S_1(y_i^{(n)}) \right) \right\|_X \leq \varepsilon \cdot \mathbf{1}.$$

By Theorem 1, we obtain that M is a relatively cyclically compact set, and therefore $S_1T \in K(X)$ (see Statement 5). And by Theorem 2

$$ST = (\mathbf{1} + \|S\|)(S_1T) \in K(X).$$

□

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