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ON GEOMETRY OF TWO DIMENSIONAL SURFACES IN FOUR DIMENSIONAL EUCLID SPACE

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Abstract

Geometry of two-dimensional surfaces in E^4 is an essential part of differential geometry and studied by many authors [4, 5, 6, 10]. In this paper, we give some surface in four dimensional Euclid space E^4 with nonzero Gauss curvature which is a orbit of the system of two vector fields. Smoothness is the smoothness of the class C^∞.

Keywords: *Two dimensional surface, Gauss curvature, singular foliation, vector field.*

Mathematics Subject Classification (2010): *53A05, 53A07.*

Introduction

In [1] T. Otsuki introduced curvatures $\lambda_1, \lambda_2, ..., \lambda_n$ for a surface Φ in a $(2+n)-$ dimensional Euclidean space E^{2+n}, defining a quadratic form in the normal space of the surface. In a suitable local frame of the normal space this quadratic form can be written in a diagonal form and the functions $\lambda_1, \lambda_2, ..., \lambda_2$ are the coefficients in the diagonalized form. These curvatures are related to the Gauss curvature K of Φ:

$$K = \lambda_1 + \lambda_2 + ..., \lambda_n.$$

For a surface Φ in the four-dimensional Euclidean space E^4 the curvatures $\lambda_1 \leq \lambda_2$ are the maximum and minimum, respectively of the Lipschitz-Killing curvature of the surface[4]. The function $\lambda_1$is called the principal curvature and the function $lambda_2$ - the secondary curvature of Φ in E^4.

Let us consider a set of vector fields $D \subset V(M)$of the Lie algebra of all smooth (class C^∞) vector fields $V(M)$ and the smallest Lie subalgebra containing D by $A(D)$. Let $t \to X^t(x)$ be an integral curve of the vector field X with the initial point x for $t = 0$, which is defined in some region $I(x)$ of real line.

Definition 1. *The orbit $L(x)$ of a system D of vector fields through a point x is the set of points y in M such that there exist $t_1, t_2, ..., t_k \in R$ and vector fields $X_{i_1}, X_{i_2}, ..., X_{i_k} \in D$ such that*

$$y = X_{i_k}^{t_k}(X_{i_{k-1}}^{t_{k-1}}(...(X_{i_1}^{t_1})...)),$$

where k is an arbitrary positive integer.

There are many investigations which devoted to the topology and geometry of orbits of a system of vector fields [3, 7]. The fundamental result in the study of orbits is Sussmann theorem [11], which asserts that every orbit is an immersed submanifold of M.

Recall that a mapping P that takes each point $x \in M$ to some subspace $P(x) \subset T_xM$ is called a distribution. If dim $P(x) = k$ for all $x \in M$, then P is called a k-dimensional distribution. A distribution P is said to be smooth if, for each point $x \in M$, there exists a neighborhood $U(x)$ of the point and smooth vector fields $X_1, X_2, ..., X_m$ defined on $U(x)$ such that the vectors

$$X_1(y), X_2(y), ..., X_m(y)$$

form a basis of the subspace $P(y)$ for each $y \in U(x)$.

A family D of smooth vector fields naturally generates the smooth distribution that takes each point $x \in M$ to the subspace $P(x)$ of the tangent space T_xM spanned by the set

$$D(x) = \{X(x) : X \in D\}.$$

Obviously, the dimension of the subspace $P(x)$ can vary from point to point.

A distribution P is said to be completely integrable if, for each point $x \in M$, there exists a connected submanifold N_x of the manifold M such that $T_yN_x = P(y)$ for all $y \in N_x$.

The submanifold N_x is callled an integral submanifold of the distribution P. For a vector field X, we write $X \in P$ if $X(x) \in P(x)$ for all $x \in M$.

A distribution P of constant dimension is said to be involutive if the inclusion $X, Y \in P$ implies that $[X, Y] \in P$, where $[X, Y]$ is the Lie bracket of the fields X and Y.

The Frobenius theorem [9] provides a necessary and suffcient condition for the completely integrability of a distribution of constant dimension.

Theorem 1 (Frobenius). *A distribution P on a manifold M is completely integrable if and only if it is involutive.*

Let $A(D)$ be the smallest Lie algebra containing the set D. By setting $A_x(D) = \{X(x) : X \in A(D)\}$, we obtain an involutive distribution $P_D : x \to Ax(D)$. If the dimension $dimA_x(D)$ is independent of x, then the distribution $P_D : x \to Ax(D)$ is completely integrable by the Frobenius theorem.

If the dimension $dimA_x(D)$ depends on x, then, as examples show, the distribution $P_D : x \to A_x(D)$ is not necessarily completely integrable.

The Frobenius theorem generalized by Hermann to distributions of variable dimension provides a necessary and sufficient condition for the complete integrability of distributions which is finitely generated [9].

Definition 2. *A system of vector fields*

$$D = \{X_1, X_2, ..., X_k\}$$

on M is in involution if there exist smooth real-valued functions $f_{ij}^{l}(x), x \in M, i,j,l = 1,...,k$ such that for each (i,j) it takes

$$[X_i, X_j] = \sum_{l=1}^{k} f_{ij}^{l}(x) X_l.$$

Theorem 2 (Hermann [9])**.** *A system of smooth vector fields*

$$D = \{X_1, X_2, ..., X_k\}$$

on M is completely integrable if and only if it is involutive.

1 On the Geometry of Two Dimensional Surface in the Four-Dimensional Euclidean Space

First of all let us recall notion of singular foliation [12].

Definition 3. *A subset L of M is said to be a $k-leaf$ of M if there exists a differentiable structure σ on L such that*

(i) (L,σ) is a connected k-dimensional immersed submanifold of M, and

(ii) if N is an arbitrary locally connected topological space,and $f : N \to M$ is a continuous function such that $f(N) \subset L$, then $f : N \to (L,\sigma)$ is continuous.

It follows from the properties of immersions that if $f : N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$, then $f : N \to (L,\sigma)$ is also differentiable.In particular, σ is the unique differentiable structure on L which makes L into an immersed k-dimensional submanifold of M.

Since M is paracompact,every connected immersed submanifold of M is separable,and so the dimensional k of a leaf L is uniquely determined.

Definition 4. *We say that $\boldsymbol{F}$ is a singular C^q-foliation of M if $\boldsymbol{F}$ is partition of M into C^q-leaves of M such that,for every $x \epsilon M$, there exists a local C^q-chart ψ of M with the following properties:*

(a) The domain of ψ is of the from $U \times W$, where U is an open neighborhood of 0 in R^k, W is an open neighborhood of 0 in R^{n-k}, and k is the dimension of the leaf through x.

(b) $\psi(0,0) = x$.

(c) If L is a leaf of $\boldsymbol{F}$, then $L \cap \psi(U \times W) = \psi(U \times l)$, where $l = \{w \epsilon W : \psi(0,w) \epsilon L\}$.

A leaf dimension of which is maximal is called regular otherwise it is called singular.

It is known that orbits of a family of vector fields generate singular foliation [11], [12]. There are many investigations which devoted to the topology and geometry of singular foliations [3, 7].

In this paper we study the geometry of the singular foliation which generated by orbits of two vector fields.

Let us consider a family of $D = \{X, Y\}$ vector fields on four-dimensional Euclidean space E^4 with Cartesian coordinates x_1, x_2, x_3, x_4, where

$$X = e^{x_1}\frac{\partial}{\partial x_1} + x_4 e^{x_1}\frac{\partial}{\partial x_4}, \tag{1}$$

$$Y = (x_2^2 - x_3^2)\frac{\partial}{\partial x_2} + 2x_2x_3\frac{\partial}{\partial x_3} - 4x_2x_4\frac{\partial}{\partial x_4}. \tag{2}$$

Theorem 3. *The family of orbits of the vector fields (1),(2) generates a singular foliation whose regular leaf is a surface with nonzero Gauss curvature.*

Proof. Now we check the condition of Hermann theorem. It is easy to check that $[X_1, X_2] = 0$. It follows from Hermann theorem the family D is completely integrable.

We need to find the invariant functions of the groups generated by vector fields (1),(2). It is not difficult to check that the functions

$$F^1(x_1, x_2, x_3, x_4) = x_3^2 x_4 e^{-x_1}, F^2(x_1, x_2, x_3, x_4) = x_3 + \frac{x_2^2}{x_3} \tag{3}$$

are invariant functions. It is known that a function f is a invariant function if and only if $X(f) = 0$ [9] We can check that it holds the following equalities

$$X(F^1) = 0, X(F^2) = 0, Y(F^1) = 0, Y(F^2) = 0. \tag{4}$$

These invariant functions give us a family of two-dimensional surfaces

$$\begin{cases} x_3^2 x_4 e^{-x_1} = C_1 \\ x_3 + \frac{x_2^2}{x_3} = C_2 \end{cases} \tag{5}$$

where C_1, C_2 are constants.

For given C_1, C_2 let us denote by F^C the connectivity component of the regular surface, which is defined by the system of equations (5).

For definiteness, we will assume that $C_1 > 0$. If $p^0(x_1^0, x_2^0, x_3^0, x_4^0) \in F^C$, it follows from equalities (4) the orbit $L(p^0)$ is contained in the surface F^C.

At any point $p(x_1, x_2, x_3, x_4)$ of the F^C vectors $X(p), Y(p)$ linearly independent which shows the orbit $L(p^0)$ is a two-dimensional manifold. It follows every orbit is an open subset of a surface F^C. Different orbits do not intersect, therefore, due to the connectivity of the surface F^C, they coincide. It follows that $L(p^0) = F^C$.

Now one can check the metric characteristics of the surface F^C. In order to find Gauss curvature we will use formulas from the paper [2].

We need to find gradient vectors

$$\begin{aligned} gradF^1 &= \{-x_3^2 x_4 e^{-x_1}, 0, 2x_3x_4e^{-x_1}, x_3^2 e^{-x_1}\} \\ gradF^2 &= \{0, \frac{2x_2}{x_3}, 1 - \frac{x_2^2}{x_3^2}, 0\}. \end{aligned} \tag{6}$$

At regular point $p(x_1, x_2, x_3, x_4)$ bivector

$$[gradF^1, gradF^2]$$

is nonzero vector and its length equal to

$$\Delta = |gradF^1|^2 \cdot |gradF^2|^2 - \langle gradF^1, gradF^2 \rangle^2.$$

Note that $\Delta > 0$ due to the regularity of the surface F^C.

Let us introduce the notation

$$F_{\alpha i} = \frac{\partial F_\alpha}{\partial x_i}, F_{\alpha ij} = \frac{\partial^2 F_\alpha}{\partial x_i \partial x_j}$$

Then we define vectors

$$\sigma_{rs} = F^1_{rs} gradF^2 - F^2_{rs} gradF^1$$

and matrix

$$M_{rlsn} = \frac{[(\sigma_{rs}\sigma_{ln}) - (\sigma_{rn}\sigma_{ls})]}{4\Delta}$$

Let's order all pairs of different indices (i, j) in the following order

$$(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3.4).$$

There are six pairs in this line. We number these pairs from 1 to 6. For example, 3 corresponds to the pair (1,4).

Now we define symmetric matrix $A = (A_{\alpha\beta})$, where $A_{\alpha\beta} = M_{iljk}$, α is the number of the pair (i, l), β is the number of the pair (j, k).

Components of the bivector d have following form

$$d^{ij} = \begin{vmatrix} F^1_i & F^1_j \\ F^2_i & F^2_j \end{vmatrix}.$$

Let q be a bivector complementary to d whose components are $q^{ij} = \varepsilon^{ijkl} d^{kl}$, where ε^{ijkl} is Kronecker symbol.

Now we are ready to write formula for Gauss curvature [2]:

$$K_G = \frac{A_{\alpha\beta} q^\alpha q^\beta}{|q|^4}$$

Let us write the matrix A in expanded form. The elements of this matrix are M_{ijkl}, but we will only indicate the indices of the elements. We have

$$\mathbf{A} = \begin{pmatrix} (1212) & (1213) & (1214) & (1223) & (1224) & (1234) \\ (1312) & (1313) & (1314) & (1323) & (1324) & (1334) \\ (1412) & (1413) & (1414) & (1423) & (1424) & (1434) \\ (2312) & (2313) & (2314) & (2323) & (2324) & (2334) \\ (2412) & (2413) & (2414) & (2423) & (2424) & (2434) \\ (3412) & (3413) & (3414) & (3423) & (3424) & (3434) \end{pmatrix}$$

Now we can calculate Gauss curvature:

$$\Delta = \Big[(x_4^2(x_3^2+2)+x_3^2)(x_2^2+x_3^2)-4x_3^2x_4^2(x_3^2-x_2^2)\Big]x_3^{-2}e^{-2x_1}$$

$$q=\{\frac{x_2^2-x_3^2}{e^{x_1}},\frac{2x_2x_3}{e^{x_1}},-\frac{4x_2x_4}{e^{x_1}},0,\frac{x_4(x_3^2-x_2^2)}{e^{x_1}},-\frac{2x_2x_3x_4}{e^{x_1}}\}$$

$$|q|^4=e^{-4x_1}\left((x_2^2+x_3^2)^2+4x_2^2x_4^2(4+x_3^2)+x_4^2(x_3^2-x_2^2)^2\right)^2$$

At least we can write Gauss curvature in the following form

$$K_G=\frac{x_4^2\Big((x_3^2-x_2^2)(3(x_3^2-x_2^2)+4x_2^2x_3)+2x_2^2((x_2^2+x_3^2)^2-2x_4^2)\Big)}{\Delta\Big((x_2^2+x_3^2)^2+x_4^2(4x_2^2(4+x_3^2)+(x_3^2-x_2^2)^2)\Big)^2}. \tag{7}$$

The formula shows that at regular points Gauss curvature is nonzero.

Note the orbit passing through the origin is a singular leaf. It is one dimensional leaf. The values of functions

$$\begin{cases} x_3^2x_4e^{-x_1} \\ x_3+\frac{x_2^2}{x_3} \end{cases} \tag{8}$$

at the origin are equal zero.

At the points of this orbit vectors

$$gradF^1=\{-x_3^2x_4e^{-x_1},0,2x_3x_4e^{-x_1},x_3^2e^{-x_1}\}$$
$$gradF^2=\{0,\frac{2x_2}{x_3},1-\frac{x_2^2}{x_3^2},0\}. \tag{9}$$

are not linearly independent i.e. the are collinear. Therefore, the bivector

$$[gradF^1,gradF^2]$$

is the zero vector and its length equal to zero: $\Delta=0$. Therefore, at the points of the singular leaf formula (7) is undefined.

□

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