[Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences](https://bulletin.nuu.uz/journal)

Manuscript 1263

Boundary value problem with the Bitsadze-Samarsky condition for a loaded equation of parabolic-hyperbolic type in a doubly connected region

Bozor Islomov

Oybek Yunusov

Follow this and additional works at: https://bulletin.nuu.uz/journal

Part of the Physical Sciences and Mathematics Commons

BOUNDARY VALUE PROBLEM WITH THE BITSADZE-SAMARSKY CONDITION FOR A LOADED EQUATION OF PARABOLIC-HYPERBOLIC TYPE IN A DOUBLY CONNECTED REGION

Bozor Islomov, Oybek Yunusov
National University of Uzbekistan, Tashkent, Uzbekistan
e-mail: islomovbozor@yandex.com, oybek198543@mail.ru

Abstract

This paper is devoted to the study of a nonlocal boundary value problem for a loaded equation of parabolic-hyperbolic type in a special domain.Using representations of the general regular solution, are proven the unique solvability of the problem posed.

Keywords: *loaded equation, nonlocal boundary value problem, Mixed type equation, Parabolic-hyperbolic type, general solution representation, integral Volterra equation with shift, fractional derivatives.*

Mathematics Subject Classification (2010): *35M10, 35M12, 35K10, 35K65, 35L80, 45D05.*

Introduction

Since the fifties of the last century, researchers have been intensively studying local and nonlocal problems for partial differential equations in simply connected regions. Currently, the range of such tasks is expanding in various directions. Among them, problems with displacement occupy a special place. This is explained by the fact that such problems cover various correct local boundary value problems, and many problems are reduced to them, for example, biological synergetics, genetics, immunology, transonic gas dynamics and thermal physics.

One of the most important classes of unloaded partial differential equations are second-order equations parabolo - hyperbolic and elliptic - hyperbolic types.An analogue of the Tricomi problem in a doubly connected domain for an equation of mixed type with one line of degeneracy was first posed and studied in A.V. Bitsadze [1, p. 30], and for the Lavrentiev–Bitsadze equation - in the works of M. S. Salakhitdinov and A. K. Urinov [2].

Boundary value problems in classical domains for a loaded equation of hyperbolic type were studied in A.M.Nakhushev [3], V.M.Kaziev [4], A.Kh.Attaev [5], and the work of M.T.Dzhenaliev [6], M.Kh.Shkhankova [7] studied the local and nonlocal problem for a loaded equation of parabolic type. A.V. Borodin [8] studied the Dirichlet problem for a loaded equation of elliptic type. In the works of K.U.Khubiev [9], B.Islomov and D.M.Kuryazov [10], M.I.Ramazanov [11], analogues of the Tricomi and Gellerstedt problem for loaded hyperbolic-parabolic type equations were studied, and the Dirichlet problem for loaded equation with the Lavrentiev a– Bitsadze operator in a rectangular domain were studied in the works of K.B. Sabitova and E.P. Melisheva [12-13].

Local and non-local problems in simply connected domains for loaded equations of elliptic -hyperbolic and parabolic -hyperbolic types, when the loaded part contains a trace or derivative of the desired function, have been little studied. We note the works of B.I. Islomov and D.M. Kuryazov [14], B.I. Islomov and U.I. Boltaeva [15], [16], B.I. Islomov and Zh.A. Kholbekov [17], K.B. Sabitov [18], R.R. Ashurova and S.Z. Zhamalova [19], Yu.K. Sabitova [20], V.A. Eleeva [21], [31-36]. This is due, first of all, to the lack of representation of a general solution for such equations; on the other hand, such problems are reduced to little-studied integral equations with a shift.

Currently, the range of such tasks is expanding in various special areas. However, boundary value problems for loaded equations of mixed type with an integral operator of fractional order in doubly connected domains have still not been sufficiently studied. Note that local and nonlocal boundary value problems for a loaded equation of elliptic -hyperbolic type in a doubly connected domain were studied in [22-25], in which the loaded part contains a differential operator or a trace of the desired function. The works [26-27] outline a technique for formulating correct boundary value problems with displacement for loaded second-order linear hyperbolic equations in special domains. It is shown that the correctness of such boundary value problems is significantly influenced by the loaded part of the equation under consideration.

Based on this, the present work is devoted to the formulation and study of a nonlocal boundary value problem for a loaded equation of parabolic - hyperbolic type in a special domain .

1 Statement of the problem BS_μ

Let Ω_0-be the domain bounded by segments A_jB_j, A_jK_j, B_jN_j, K_jN_j of straight lines $y=0$, $x=(-1)^{j-1}$, $x=(-1)^{j-1}q$, $y=1$ for $y>0$. $\Omega-$ is the domain bounded by segments $A_jB_j (j=1,2)$ of the Ox-axis and for $y<0$ by characteristics $A_jC_1: x-(-1)^{j-1}y=(-1)^{j-1}$, $B_jC_2: x-(-1)^{j-1}y=(-1)^{j-1}q$ of the following equations:

$$0=\begin{cases} u_{xx}-u_y-\mu_j u_x(x,0), & y>0, \\ u_{xx}-u_{yy}-\mu_{j+2}u(\xi,0), & y<0, \end{cases} \tag{1}$$

going out fromp points $A_j\left((-1)^{j-1};\,0\right)$ and $B_j\left((-1)^{j-1}q;0\right)$, and intersecting at points $C_1(0;-1)$ and $C_2(0;-q)$, $\xi=x+y$. In equation (1), q, μ_j, $\mu_{j+2} (j=1,2)$ are the given real numbers, and

$$0<q<1, \quad (-1)^j\mu_j>0, \quad (-1)^{j-1}\mu_{j+2}>0, \quad (j=1,2). \tag{2}$$

Let us introduce the following notation:

$I=\{(x,y):\ \ x=0,\ \ -1<y<-q\},\ J_1=\{(x,y):q<x<1,\ \ y=0\},$
$J_2=\{(x,y):-1<x<-q,\ \ y=0\},\ D_j,E_j\in A_jC_1,$

$$B_jD_j:\ \ x+(-1)^{j-1}y=(-1)^{j-1}q,\quad C_2E_j:x+(-1)^{j-1}y=(-1)^j q,$$

$$\left.\begin{matrix}A_jD_j\\A_jE_j\end{matrix}\right\}:\ \ x-(-1)^{j-1}y=(-1)^{j-1},\ (j=1,2).$$

Characteristic triangles $A_jB_jD_j$ and quadrangles $B_jC_2E_jD_j$,$C_2E_1C_1E_2$, $(j=1,2)$, we denote by Ω_j and Ω_{j+2}, Ω_5, respectively.

$$\Omega_{5j}=\Omega_5\cap\left\{(-1)^jx<0,\ \ y<0\right\},\Omega_{0j}=\Omega_0\cap\left\{(-1)^jx<0,\ \ y>0\right\},\Omega_0=\Omega_{01}\cup\Omega_{02},$$

$$\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4\cup\Omega_5\cup B_1D_1\cup C_2E_1\cup C_2E_2\cup B_2D_2,\Delta_j=\Omega_j\cup\Omega_{j+2}\cup\Omega_{5j},$$

$$\Delta=\Omega_0\cup\Omega\cup I\cup J_1\cup J_2,\Delta_{0j}=\Omega_{0j}\cup A_jB_j\cup K_jN_j,\ \ \Delta_j^*=\Omega_j\cup\Omega_{0j},$$

$$\theta_1(x)=\theta_2\left(\frac{x+1}{2};\ \ -\frac{x-1}{2}\right),\theta_2(x)=\theta_2\left(\frac{x-1}{2};\ \ -\frac{x+1}{2}\right),\tag{3}$$

$\theta_j(x)$ is the point of intersection of characteristic A_jD_j with the characteristic going out of point $Z_j(x,0)\in J_j$, $(j=1,2)$.

Problem BC_μ**.** Find function $u(x,y)$ that has the following properties:

1)$u\,(x,y)\in C(\bar{\Delta})\cap C^1(\Delta)$;

2)$u(x,y)\in C^{2,1}_{x,y}(\Delta_{0j})\cap C^{2,2}_{x,y}(\Delta_j)$ and satisfies equation (1) in domains Δ_{0j} and Δ_j $(j=1,2)$;

3)$u(x,y)$ satisfies the following boundary conditions:

$$u\,(x,y)|_{A_jK_j}=\varphi_j\,(y)\,,\qquad u\,(x,y)|_{B_jN_j}=g_j\,(y)\,,\quad 0\le y\le 1,\tag{4}$$

$$u\,[\theta_j\,(x)]+a_j\,(x)\,u\,(x,0)=b_j\,(x)\,,\quad (x,0)\in\bar{J}_j,\tag{5}$$

$$u(x,y)|_{B_jC_2}=\psi_j(x),\qquad 0\le\left((-1)^{j-1}x\right)\le q,\ \ (j=1,2),\tag{6}$$

where $\varphi_j(y)$, $g_j(y)$, $\psi_j(x)$, $a_j(x)$, $b_j(x)$ $(j=1,2)$ are the given functions and

$$\psi_1(0)=\psi_2(0),\quad g_j(0)=\psi_j((-1)^{j-1}q),\quad \varphi_j(0)(1+a_j(-1)^{j-1})=b_j((-1)^{j-1}),\tag{7}$$

$$\varphi_j\,(y)\,,\quad g_j\,(y)\in C\,[0,1]\cap C^1(0,1),\tag{8}$$

$$a_j(x),\ b_j(x)\ \in C^1\left(\bar{J}_j\right)\cap C^3\,(J_j)\,,\tag{9}$$

$$\psi_j(x)\in C\left(0\le\left((-1)^{j-1}x\right)\le q\right)\cap C^2\left(0<\left((-1)^{j-1}x\right)<q\right).\tag{10}$$

2 Investigation of the problem BS_μ

To study problem BC_μ, the following problem plays an important role.

Task $BS_{j\mu}$. Find function $u(x,y)$ that has the following properties:

1)$u\,(x,y) \in C(\bar{\Delta}_j^*) \cap C^1(\Delta_j^* \cup J_1 \cup J_2)$;

2) $u(x,y) \in C^{2,1}_{x,y}(\Omega_{0j}) \cap C^{2,2}_{x,y}(\Omega_j)$ and satisfies equation (1) in domains Ω_{0j} and Ω_j $(j=1,2)$;

3) $u(x,y)$ satisfies boundary conditions (4) and (5).

Theorem 1. If conditions (2), (7)-(10) and

$$(-1)^j\,(1+2a_j(x)) \geq 0, \quad a_j'(x) \geq 0, \;\; \forall x \in \bar{J}_j, \;\; (j=1,2), \tag{11}$$

are met, then there is a unique regular solution to problem $C_{j\mu}$ in domain Ω .

Proof. Solution to the Cauchy problem with the following conditions

$$u\,(x,-0) = \tau_j\,(x)\,,\; (x,0) \in \bar{J}_j, \quad u_y\,(x,-0) = \nu_j\,(x)\,,\; (x,0) \in J_j,\; (j=1,2) \tag{12}$$

for equation (1) in domain Ω_j has the following form:

$$u\,(x,y) = \frac{1}{2}\left[\tau_j\,(x+y) + \tau_j\,(x-y)\right] + \frac{1}{2}\int_{x-y}^{x+y} \nu_j\,(\xi)\,d\xi +$$

$$+\frac{\mu_{j+2}}{4}\int_{x-y}^{x+y}(x-y-\xi)\tau_j\,(\xi)\,d\xi,\; (j=1,2). \tag{13}$$

Substituting (13) for $j=1$ in (5) and considering

$$u\left[\theta_1\,(x)\right] = u\left[\frac{x+1}{2};\;\; \frac{x-1}{2}\right] = \frac{1}{2}\left[\tau_1(x)+\tau_1(1)\right] - \frac{1}{2}\int_x^1 \nu_1(t)dt - \frac{\mu_3}{4}\int_x^1 (1-t)\tau_1\,(t)\,dt$$

we obtain:

$$\frac{1}{2}\left[\tau_1(x)+\tau_1(1)\right] - \frac{1}{2}\int_x^1 \nu_1(t)dt - \frac{\mu_3}{4}\int_x^1 (1-t)\tau_1\,(t)\,dt + a_1(x)\tau_1(x) = b_1(x). \tag{14}$$

Differentiating (14) with respect to x, we obtain a functional relationship between $\tau_1(x)$ and $\nu_1(x)$ brought from domain Ω_1 to J_1:

$$\left[1+2a_1(x)\right]\tau_1'(x) + \left[\frac{\mu_3}{2}(1-x) + 2a_1'(x)\right]\tau_1(x) + \nu_1(x) = 2b_1'(x), \;\; (x,0) \in J_1. \tag{15}$$

Similarly, using the solution to the Cauchy problem (13) for $j=2$ with initial conditions (12) for equation (1) in domain Ω_2 considering (3) and (5), we obtain a functional relationship between $\tau_2(x)$ and $\nu_2(x)$, brought from domain Ω_2 to J_2:

$$\left[1+2a_2(x)\right]\tau_2'(x) + 2a_2'(x)\tau_2(x) - \frac{\mu_4}{2}\int_{-1}^{x} \tau_2(t)dt - \nu_2(x) = 2b_2'(x), \quad (x,0) \in J_2. \tag{16}$$

Consequently, as in [28, pp. 40-47], due to conditions 1) - 2) of the problem $BS_{j\mu}$, passing to the limit at $y \to +0$ in equation (1) considering (12), we obtain a functional relationship between $\tau_j'(x)$ and $\nu_j(x)$ brought from domain Ω_{0j} to J_j:

$$\tau_j'(x) - \mu_j \int_{(-1)^{j-1}}^{x} \tau_j'(t)\,dt = \tau_j'((-1)^{j-1}) + \int_{(-1)^{j-1}}^{x} \nu_j(t)\,dt, \tag{17}$$

where $\tau_j'((-1)^{j-1})$ $(j = 1, 2)-$ is the unknown constant to be determined.

Eliminating $\nu_j(x)$ from relations (15), (16) and (17), after some calculations, we obtain the following integral equation for $\tau_j'(x)$ $(j = 1, 2)$:

$$\tau_j'(x) - \int_{(-1)^{j-1}}^{x} M_j(x,t)\tau_j'(t)\,dt = \tau_j'((-1)^{j-1}) + F_j(x), \quad (x,0) \in \bar{J}_j, \tag{18}$$

where

$$M_1(x,t) = \mu_1 - (1 + 2a_1(x)) - \int_x^t \left[2a_1'(z) + \frac{\mu_3(1-z)}{2}\right] dz, \tag{19}$$

$$M_2(x,t) = \mu_2 + 1 + 2a_2(x) + \int_t^x \left[2a_2'(z) - \frac{\mu_4(z-t)}{2}\right] dz, \tag{20}$$

$$F_1(x) = -\varphi_1(0) \int_x^1 \left(\frac{\mu_3(1-t)}{2} + 2a_1(t)\right) dt - 2\int_x^1 b_1'(t)dt, \tag{21}$$

$$F_2(x) = \varphi_2(0) \int_{-1}^x \left[2a_2(t) - \frac{\mu_4(1+t)}{2}\right] dt - 2\int_{-1}^x b_2'(t)dt. \tag{22}$$

By (2) and (9) from (19) - (22), it follows that

$$|M_1(x,t)| \le c_1 \quad \text{for any} \quad q \le x \le 1, \quad q \le t \le 1, \tag{23}$$

$$|M_2(x,t)| \le c_2 \quad \text{for any} \quad -1 \le x \le -q, \quad -1 \le t \le -q, \tag{24}$$

$$F_j(x) \in C\left(\bar{J}_j\right) \cap C^2\left(J_j\right), \ \ (j = 1, 2). \tag{25}$$

Thus, by (23), (24) and (25), equation (18) is a Volterra integral equation of the second kind. According to Volterra's theory of integral equations [26], we conclude that integral equation (18) is uniquely solvable in class $C\left(\bar{J}_j\right) \cap C^2\left(J_j\right)$ and its solution is given by the following formula:

$$\tau_j'(x) = \tau_j'((-1)^{j-1}) + F_j(x) + \int_{(-1)^{j-1}}^{x} \left[\tau_j'((-1)^{j-1}) + F_j(t)\right] M_j^*(x,t)dt, \tag{26}$$

where $M_j^*(x,t)$ $(j = 1, 2)-$ is the resolvent of kernel $M_j(x,t)$.

Integrating (26) from x to 1 for $j = 1$(from -1 to x for $j = 2$) with $\tau_1(1) = \varphi_1(0)$ $(\tau_2(-1) = \varphi_2(0))$, we obtain:

$$\tau_1(x) = \varphi_1(0) - \left[1 - x - \int_x^1 dt \int_t^1 M_1^*(t,z)dz\right] \tau_1'(1) - \int_x^1 F_1(t)dt +$$

$$+\int_x^1 dt\int_t^1 M_1^*(t,z)F_1(z)dz,\qquad (x,0)\in\bar{J}_1, \tag{27}$$

$$\tau_2(x)=\varphi_2(0)+\left[1+x+\int_{-1}^{x} dt\int_{-1}^{t} M_2^*(t,z)dz\right]\tau_2'(-1)+\int_{-1}^{x}F_2(t)dt-$$

$$-\int_{-1}^{x} dt\int_{-1}^{t} M_2^*(t,z)F_2(z)dz,\qquad (x,0)\in\bar{J}_2, \tag{28}$$

Now assuming in (27) and (28) $x=q$ and $x=-q$ respectively, and considering $\tau_j((-1)^{j-1}q)=g_j(0)$ $(j=1,2)$ we find the unknown constant $\tau_j'((-1)^{j-1})$:

$$\tau_j'((-1)^{j-1})=\left[(-1)^{1-j}(1-q)-\int_q^1 dt\int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)dz\right]^{-1}\times$$

$$\times\left[\varphi_j(0)-g_j(0)-\int_q^1 F_j((-1)^{j-1}t)dt+\right]$$

$$\left[+\int_q^1 dt\int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)F_j((-1)^{j-1}z)dz\right],\ (j=1,2). \tag{29}$$

By (2), (11), from (19) and (20), it follows that $(-1)^{j-1}M_j(x,t)<0,\ \forall\ x,t\in\bar{J}_j,(j=1,2)$. Consequently, the resolvent of kernel $M_j^*(x,t)$ is also negative, i.e, $(-1)^{j-1}M_j^*(x,t)<0,\ \forall\ x,t\in\bar{J}_j(j=1,2)$.This means that the denominator of formula (29) for any $q\le(-1)^{j-1}x\le 1$, $q\le(-1)^{j-1}t\le 1$ does not vanish, i.e, $(-1)^{1-j}(1-q)-\int_q^1 dt\int_t^1 M_j^*((-1)^{j-1}t;(-1)^{j-1}z)dz>0$.

By (23) - (25) from (27) and (28) considering (29), we conclude that

$$\tau_j(x)\in C^1\left(\bar{J}_j\right)\cap C^3\left(J_j\right),\ (j=1,2). \tag{30}$$

Substituting (30) into (15) and (16), and considering (9), (30), we define function $\nu_j(x)$ from class $\nu_j^-(x)\in C(\bar{J}_j)\cap C^1(J_j)$.

Thus, the solution to problem $BS_{j\mu}$ can be restored in domain Ω_{0j} as a solution to the first boundary value problem for equation (1) [28, pp. 99], and in domains $\Omega_j(j=1,2)$ as a solution to the Cauchy problem for equation (1) (see (13)).

Therefore, problem $BS_{j\mu}$ is uniquely solvable.

□

Now, to restore the solution to problem BS_μ in domains Ω_{j+2} and Ω_5, we will solve the Goursat problem for equation (1).

Problem Γ_j $(j=1,2)$. Find function $u(x,y)$ with the following properties:

1) $u(x,y)\in C(\bar{\Omega}_{j+2})\cap C^1(\Omega_{j+2})$;

2) $u(x,y)$- is a twice continuously differentiable solution to equation (1) in domain $\Omega_{j+2}\ (j=1,2)$;

3) $u(x,y)$ satisfies boundary conditions (6) and

$$u(x,y)|_{B_jD_j}=w_j(x),\qquad q\le(-1)^{j-1}x\le\frac{1+q}{2}, \tag{31}$$

where $w_j(x)$ $(j=1,2)-$are determined from

$$w_j\left(\frac{x+(-1)^{j-1}q}{2}\right)=\frac{1}{2}\left[\tau_j(x)+\tau_j((-1)^{j-1}q)\right]+\frac{(-1)^{j-1}}{2}\int\limits_x^{(-1)^{j-1}q}\nu_j\left(t\right)dt+$$

$$+\frac{\mu_{j+2}}{4}(-1)^j\int\limits_x^{(-1)^{j-1}q}(t+q)\tau_j\left(t\right)dt+\frac{\mu_3(1+(-1)^{j-1})}{8}\int\limits_x^q(x+q)\tau_1\left(t\right)dt,\quad q\leq(-1)^{j-1}x\leq1,$$

here $\nu_j(x)$ and $\tau_j(x)$ $(j=1,2)-$ are known functions determined from (15), (16), and (27)-(29), $w_j(x)$ $(j=1,2)$ belongs to class

$$w_j(x)\in C\left(q\leq(-1)^{j-1}x\leq\frac{1+q}{2}\right)\cap C^2\left(q<(-1)^{j-1}x<\frac{1+q}{2}\right),\tag{32}$$

and $w_j((-1)^{j-1}q)=\psi_j((-1)^{j-1}q)$.

Problem Γ_3. Find function $u(x,y)$ with the following properties:

1) $u(x,y)\in C(\bar{\Omega}_5)\bigcap C^1(\Omega_5)$, and function $u_x(0,y)$ can go to infinity of order less than one at the ends of interval I;

2) $u(x,y)$- is a twice continuously differentiable solution to equation (1) in domain Ω_{5j} $(j=1,2)$;

3) $u(x,y)$ satisfies the following boundary conditions:

$$u(x,y)|_{C_2E_j}=h_j(y),\qquad-\frac{1+q}{2}\leq y\leq-q,\tag{33}$$

where $h_j(y)$ $(j=1,2)-$ are determined from

$$h_1(y)=w_1\left(-y\right)-\frac{\mu_3(y+q)}{2}\int_q^{-q}\tau_1(t)dt+\psi_1\left(0\right)-\psi_1\left(q\right),$$

$$h_2(y)=w_2\left(y\right)+\frac{\mu_4 q}{2}\int_{-q}^{2y+q}\tau_2(t)dt+\psi_2\left(0\right)-\psi_2\left(-q\right),$$

and belong to class

$$h_j(y)\in C\left[-\frac{1+q}{2},-q\right]\cap C^2\left(-\frac{1+q}{2},-q\right),\tag{34}$$

and $h_1(-q)=h_2(-q)$ $\Rightarrow\psi_1(0)=\psi_2(0)$, here $\psi_j(x)$, $w_j(x)$, $\tau_j(x)$ - are known functions determined from (6), (12), (27), (28) and (29) respectively.

Theorem 2. If conditions (2), (10), (30), (32) are satisfied, then the solution to problem Γ_j exists and is unique in domain Ω_{j+2}.

Proof. The general solution to equation (1) in domain Ω_{j+2} has the following form [30, pp. 77]:

$$u\left(x,y\right)=\Phi_j\left(x+y\right)+G_j\left(x-y\right)+$$

$$\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt, \quad (j=1,2), \tag{35}$$

where $\Phi_j(x)$, $G_j(x)-$ are arbitrary twice continuously differentiable functions, and function $\tau_j\left(x\right)$ $(j=1,2)$ is determined from (27), (28) and (39).

Substituting (35), into (6) and (31), we find

$$u\left(x,y\right)=\psi_j\left(\frac{x+y+(-1)^{j-1}q}{2}\right)+w_j\left(\frac{x-y+(-1)^{j-1}q}{2}\right)-\psi_j((-1)^{j-1}q)+$$

$$+\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt, \quad (j=1,2). \tag{36}$$

Considering the properties of functions $w_j(x),\psi_j(x)$ and $\tau_j(x)$, it follows from (36) that the solution to problem Γ_j $(j=1,2)$ exists, is unique and belongs to class $u(x,y)\in C(\bar{\Omega}_{j+2})\cap C^2(\Omega_{j+2})$.

Therefore, problem Γ_j $(j=1,2)$ is uniquely solvable.

□

Theorem 3. If conditions (2), (34) are satisfied, then in domain Ω_5 the solution to problem Γ_3 exists and is unique.

To prove Theorem 3, the following problems play an important role:

Problem Γ_{3j} $(j=1,2)$. Find in domain Ω_{5j} a solution $u(x,y)\in C(\bar{\Omega}_{5j})\cap C^1(\Omega_{5j}\cup I)\cap C^2(\Omega_{5j})$ to equation (1) that satisfies conditions (33) and

$$u_j\left(0,y\right)=\tau_3\left(y\right), \qquad (0,y)\in\bar{I}, \tag{37}$$

where $\tau_3\left(y\right)$ - is a given function, and

$$\tau_3\left(y\right)\in C\left(\bar{I}\right)\cap C^2\left(I\right), \qquad \tau_3\left(-q\right)=h_1\left(-q\right)=h_2\left(-q\right). \tag{38}$$

Let us study problem Γ_{3j} $(j=1,2)$. General solution to equation (1) in domain Ω_{5j} has the following form [15], [30, pp. 135-137]:

$$u\left(x,y\right)=\Phi_{j+2}\left(x+y\right)+G_{j+2}\left(x-y\right)+$$

$$\frac{\mu_{j+2}(x-y-(-1)^{j-1}q)}{4}\int_{(-1)^{j-1}q}^{x+y}\tau_j\left(t\right)dt, \quad (j=1,2), \tag{39}$$

where $\Phi_{j+2}(x)$, $G_{j+2}(x)$ are arbitrary twice continuously differentiable functions, and function $\tau_j\left(x\right)$ $(j=1,2)$-is determined from (27), (28) and (29).

Then, substituting (39) into (33) and (37), we find a solution to problem Γ_{31} and Γ_{32} for equation (1) in domains Ω_{51} and Ω_{52}:

$$u\left(x,y\right)=\tau_3\left(x+y\right)-h_1\left(\frac{x+y-q}{2}\right)+h_1\left(\frac{y-x-q}{2}\right)+\frac{\mu_3 x}{2}\int_{-q}^{x+y}\tau_1\left(t\right)dt, \tag{40}$$

and

$$u\left(x,y\right)=\tau_3\left(y-x\right)+h_2\left(\frac{x+y-q}{2}\right)-h_2\left(\frac{y-x-q}{2}\right)+$$

$$+\frac{\mu_4(x-y+q)}{4}\int_{-q}^{y-x}\tau_2(t)\,dt+\frac{\mu_4(x-y-q)}{4}\int_{-q}^{x+y}\tau_2(t)\,dt, \qquad (41)$$

respectively.

By (30), (34), (38) from (40) and (41), we conclude that the solution to problem Γ_{3j} $(j=1,2)$exists, is unique, and belongs to class $u(x,y)\in C(\bar{\Omega}_{5j})\cap C^1(\Omega_{5j}\cup I)\cap$ $\cap C^2(\Omega_{5j})$.

Now let us proceed to problem Γ_3. Differentiating (40) and (41) with respect to x, then tending x to zero, we obtain a functional relationship between $\nu_3(y)$ and $\tau_3'(y)$, brought from domains Ω_{51} and Ω_{52} to I:

$$\nu_3(y)+(-1)^j\tau_3'(y)=F_{j+2}(y), \quad -1<y<-q, \quad (j=1,2), \qquad (42)$$

where

$$\nu_3(y)\equiv u_x(-0,y)=u_x(+0,y), \ \ (0,y)\in I, \qquad (43)$$

$$F_3(y)=-h_1'\left(\frac{y-q}{2}\right)+\frac{\mu_3}{2}\int_{-q}^{y}\tau_1(t)dt,$$

$$F_4(y)=-h_2'\left(\frac{y-q}{2}\right)-\frac{\mu_4 q}{2}\tau_2(y)+\frac{\mu_4}{2}\int_{-q}^{y}\tau_2(t)dt. \qquad (44)$$

Eliminating function $\nu_3(y)$ from (42) considering (43), we obtain

$$\tau_3'(y)=F_4(y)-F_3(y), \qquad (45)$$

Integrating (45) from $-q$ to y considering $\tau_3(-q)=h_1(-q)$, we obtain:

$$\tau_3(y)=\int_{-q}^{y}\left[F_4(t)-F_3(t)\right]\,dt+h_1(-q). \qquad (46)$$

By (30), (34) from (46) it follows that $\tau_3(y)\in C\left(\bar{I}\right)\bigcap C^2(I)$. Consequently, after defining function $\tau_3(y)$, problem Γ_3 is reduced to the study of problems Γ_{31} and Γ_{32}, where $\tau_3(y)$- is determined by formula (46).

Hence, we conclude that problem Γ_3 is uniquely solvable.

Theorem 3 is proven.

This completes the study of problem BS_μ for equation (1).

3 Conclusion

This work is devoted to the statement and study of boundary value problem with the Bitsadze-Samarskii condition for a loaded equation of parabolic-hyperbolic type in a doubly connected domain and related results on the existence and behavior of solutions to the problem with integral gluing conditions. Thus, we can study various boundary value problems for loaded equations of generalized parabolic and hyperbolic types with fractional operators.

References

[1] Bitsadze A.V. On the problem of equations of mixed type. Tr. MIAN USSR. Vol. 41, pp. 3-59 (1953).

[2] Salakhitdinov M.S., Urinov A.K. Nonlocal boundary value problem in a doubly connected domain for an equation of mixed type with nonsmooth lines of degeneracy. Dokl. Academy of Sciences of the USSR. Vol. 299, No. 1, pp. 63-66 (1988).

[3] Nakhushev A.M. Loaded equations and their applications. Differential equations. Vol. 19, No. 1, pp. 86-94 (1983).

[4] Kaziev V.M. On the Darboux problem for one loaded second-order integro-differential equation. Differential equations. Vol. 14, N. 1, pp. 181-184 (1978).

[5] Attaev A.Kh. Boundary value problems for the loaded wave equation. Differential equations: Abstract. report regional interuniversity seminar. May 20-25, Kuibishev, pp. 9-10 (1984).

[6] Dzhenaliev N.T. On a boundary value problem for a linear loaded parabolic equation with nonlocal and boundary conditions. Differential equations. Vol. 27, No. 10, pp. 1925-1927 (1991).

[7] Shkhanukov M.Kh. Difference method for solving one loaded equation of parabolic type. Differential equations. Vol. 13, No. 1, pp. 163-167 (1977).

[8] Borodin A.B. About one estimate for elliptic equations and its application to loaded equations. Differential equations. Vol. 13, No.1, pp. 17-22 (1977).

[9] Khubiev K.U. Gellerstedt problem for a loaded equation of mixed type with data on non-parallel characteristics. Dokl. Adyg. inter. academic. sciences. No. 1, pp. 1-4 (2008).

[10] Islomov B., Kuryazov D.M. Boundary value problems for a mixed loaded equation of third order of parabolic-hyperbolic type. Uzbek mathematical journal. No. 2, pp. 29-35 (2000). [in Russian]

[11] Ramazanov M.I. On a nonlocal problem for a loaded hyperbolic-elliptic type in a rectangular domain. Mathematical journal. Vol. 2, No. 4, pp. 75-81 (2002). [in Russian]

[12] Sabitov K.B., Melisheva E.P. The dirichlet problem for a loaded mixed-type equation in a rectangular domain. Russian Mathematics (Izvestiya VUZ. Matematika). Vol. 57, Issue 7, pp. 53-65 (2013).

[13] Melisheva E.P. Dirichlet problem for the loaded Lavrentiev–Bitsadze equation Vestn. Myself. state tech. univ. Phys.-math. Sciences Ser. Vol. 80, No. 6, pp. 39-47 (2010).

[14] Islomov B., Kuryazov D.M. On a boundary value problem for a loaded second-order equation. Doclady RUz. No. 1-2, pp. 3-6 (1996). [in Russian]

[15] Islomov B., Baltaeva U.I. Boundary value problems for loaded differential equations of hyperbolic and mixed types of third order. Ufa Mathematical Journal. Vol. 3, No. 3, pp. 15-25 (2011).

[16] Islomov B., Baltaeva U.I. Boundary value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. Electronic Journal of Differential Equations. No. 221, pp. 1-10 (2015).

[17] Islomov B.I., Kholbekov Zh.A. On a nonlocal boundary-value problem for a loaded parabolic-hyperbolic equation with three lines of degeneracy. Vestnik Samarskogo Gosudarstvennogo Tekhnicheskogo Universiteta, Seriya Fiziko-Matematicheskie Nauki. Vol. 25, No. 3, pp. 407-422 (2021).

[18] Sabitov K.B. Initial-boundary problem for a parabolic-hyperbolic equation with loaded terms. Russian Mathematics (Izvestiya VUZ. Matematika). Vol. 59, Issue 6, pp. 23-33 (2015).

[19] Dzhamalov S.Z., Ashurov R.R. On a nonlocal boundary-value problem for second kind second-order mixed type loaded equation in a rectangle. Uzbek Mathematical Journal. No. 3, pp. 63-72 (2018).

[20] Sabitova Yu.K. Dirichlet problem for the Lavrentiev-Bitsadze equation with loaded terms. News of universities. Mathematics. No. 9, pp. 42-58 (2018).

[21] Eleev V.A. On some boundary value problems for mixed loaded equations of the second and third order. Differential equations. Vol. 30, No. 2, pp. 230-237 (1994).

[22] Abdullayev O.Kh. About a method of research of the non-local problem for the loaded mixed type equation in double-connected domain. Bulletin KRASEC. Phys. Math. Sci. Vol. 9, No. 2, pp. 3-12 (2014).

[23] Abdullaev O.Kh. Boundary value problem for a loaded equation of elliptic-hyperbolic type in a doubly connected domain. Bulletin of KRAUNC. Phys. Math. Sciences. No. 1(8), pp. 33-48 (2014).

[24] Islomov B.I., Abdullaev O.Kh. A boundary value problem of the type of the Bitsadze problem for a third-order equation of elliptic-hyperbolic type in a doubly connected domain. Reports of the Adyghe (Circassian) International Academy of Sciences. Vol. 7, No. 1, pp. 42-46 (2004). [in Russian]

[25] Abdullaev O.Kh. Nonlocal problem for a loaded equation of mixed type with an integral operator. Vestn. Myself. state tech. un-ta. Ser. Phys. Math. Sciences. Vol. 20, No. 2, pp. 220-240 (2016).

[26] Islamov B.I., Yunusov O.M. Nonlocal boundary value problem for a loaded equation of hyperbolic type in a special domain. DAN Republic of Uzbekistan. No. 4, pp. 8-12 (2016). [in Russian]

[27] Yunusov O.M. Nonlocal boundary value problem for a loaded equation of hyperbolic type. Proceedings of the international conference Differential equations and related problems. Russia, pp. 152-155 (2018). [in Russian]

[28] Juraev T.D. Boundary value problems for equations of mixed and mixed-composite type. Tashkent, Fan. 240 p. (1979). [in Russian].

[29] Mikhlin S.G. Lectures on linear integral equations. Moscow, Fizmatgiz, 232 p. (1959).

[30] Tikhonov A.N., Samarsky A.A. Equations of mathematical physics. Moscow, Science, 736 p. (1977).