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An analogue of Hartogs lemma for separately harmonic functions with variable radius of harmonicity

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AN ANALOGUE OF HARTOGS LEMMA FOR SEPARATELY HARMONIC FUNCTIONS WITH VARIABLE RADIUS OF HARMONICITY

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Abstract

In this note we prove that if a function $u(x, y)$ is separately harmonic in a domain

$$D \times V_r = D \times \left\{ y \in \mathbb{R}^2 : |y| < r,\ r > 1 \right\} \subset \mathbb{R}^n \times \mathbb{R}^2$$

and for each fixed point $x^0 \in D$ the function $u(x^0, y)$ of variable y continues harmonically into the great circle

$$\left\{ y \in \mathbb{R}^2 : |y| < R(x^0),\ R(x^0) > r \right\},$$

then it continues harmonically into a domain

$$\left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^2 : |y| < R_*(x),\ \ x \in D \right\}$$

over a set of variables.

Keywords: *separately harmonic function, holomorphic function, P-measure, h-measure, h-pluripolar set.*

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Introduction

Suppose given two domains $D \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^m$ and two sets $E \subset D$ and $F \subset G$. Suppose that a function $u(x, y)$ is originally defined on the set $E \times F$ and possesses the following properties:

(a) for each fixed $x^0 \in E$, the function $u(x^0, y)$ can be harmonically extended to G;

(b) for each fixed $y^0 \in F$, the function $u(x, y^0)$ can be harmonically extended to D.

In this case, the above-mentioned extensions of $u(x, y)$ define a certain function on the set

$$X = (E \times G) \cup (D \times F),$$

which is called a *separately harmonic function* on X.

In the case when $E = D$, $F = G$, the function $u(x, y)$ is called separately harmonic in the domain $X = D \times G$, i.e., harmonic in each variable separately.

It is clear that the set X, in general, is not a domain. But in spite of this, the definition of separately harmonic functions on X (similarly to the definition of separately-analytic functions in complex space, see [3], [5], [9], [10], [2], [11] and [13]) makes some sense, since harmonic functions have some properties of analytic functions.

In an arbitrary domain, which in general is not represented as a product of two domains, the separately harmonic function $u(x, y)$ is defined as follows: *if the function* $u(x, y)$ *is defined in the domain* $Q \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$ $(n, m \geq 2)$ *and has the following properties*:

1) *for any* $x^0 : \{x = x^0\} \cap Q \neq \emptyset$*, the function* $u(x^0, y)$ *is harmonic in the variable* y *on the section* $\{x = x^0\} \cap Q$;

2) *for any* $y^0 : \{y = y^0\} \cap Q \neq \emptyset$*, the function* $u(x, y^0)$ *is harmonic in the variable* x *on the section* $\{y = y^0\} \cap Q$*,*

then it is called a separately harmonic function in the domain Q.

The first result about separately harmonic functions was obtained by Lelong.

Theorem 0.1 (Lelong [14])**.** *If* $u(x, y)$ *is separately harmonic function in the domain* $Q \subset \mathbb{R}^n \times \mathbb{R}^m$*, then* $u(x, y)$ *is harmonic in* Q *over the set of variables.*

A more general case, i.e., the separately harmonic function problem was studied in the works of Zeriahi [12] and Hecart [16].

1 Extremal functions in the class of harmonic functions and the Hecart theorem

In the study of harmonic function spaces, Zahariuta (see [4]) introduced the extremal function. In this section we will give the notion of H-regularity of compacts (see [15] and [16]) and the Hecart theorem on the continuation of separately harmonic functions.

Let us denote by $h(D)$ the set of all harmonic functions in the domain $D \subset \mathbb{R}^n$.

Definition 1.1 (see [4] and [15])**.** *Let* D *be a domain in* $\mathbb{R}^n$*, and* K *is a compact in* D*. Let us define*

$$\chi_0(x, K, D) := \lim_{\varepsilon \to 0} \chi_\varepsilon(x, K, D),$$

where

$$\chi_\varepsilon(x, K, D) := \overline{\lim_{y \to x}} \sup \left\{ \lambda \ln |u(y)| , u \in h(D), 0 < \lambda < \varepsilon, \|u\|_K \leq 1, \|u\|_D^\lambda \leq e \right\}.$$

Definition 1.2 (see [4] and [16])**.** *Let* $\{D_s\}_{s\in\mathbb{N}}$ *be a sequence of domains from* $\mathbb{R}^n$ *such that*

$$D_s \subset D_{s+1}, \ \bigcup_{s\in\mathbb{N}} D_s = D$$

and $\{K_r\}_{r\in\mathbb{N}}$ *be a sequence of compact subsets of* D_1 *such that*

$$K_{r+1} \Subset Int\, K_r, \ \bigcap_{r\in\mathbb{N}} K_r = K.$$

Let us define the extremal Zahariuta function $h(\cdot, K, D)$, associated with (K, D) by the formula:

$$h(x, K, D) := \overline{\lim_{y\to x}} \lim_{r\to+\infty} \chi(x, K_r, D), x \in D,$$

where

$$\chi(x, K, D) := \lim_{s\to+\infty} \chi_0(x, K, D_s).$$

Remark 1.1. *In the case $n = 2$ Zahariuta proved (see [4]) that $h(\cdot, K, D)$ is an ordinary harmonic measure of ω^*_{sh}, which is defined using subharmonic (sh) functions:*

$$\omega_{sh}(x, E, D) = \sup \left\{u(x) : u(x) \in sh(D), u|_E \leq 0, u|_D \leq 1\right\},$$

$$\omega^*_{sh}(x, E, D) = \overline{\lim_{x'\to x}} \omega_{sh}(x', E, D), x \in D.$$

It is easy to see that $\chi(\cdot, K, D) \geq \chi_0(\cdot, K, D)$ and $\chi(\cdot, K, D) \geq h(\cdot, K, D)$.

Definition 1.3 (see [4] and [15]). *A compact $K \subset \mathbb{R}^n$ is called H-regular at a point a if for any $b > 1$ there exists $M > 0$ and an open neighbourhood V of point a such that*

$$\|p\|_V \leq M\, b^n \|p\|_K, \ \forall\, p \in \mathcal{P}_n(\mathbb{R}^n), \ \forall n \in \mathbb{N},$$

where $\mathcal{P}_n(\mathbb{R}^n)$ is the vector space of all harmonic polynomials of degree at most n. A compact K is called H-regular if it is H-regular for every point $a \in K$.

H-regularity occupies a very important place in the theory of polynomial approximation of harmonic functions (see [17], [6], [7] and [8]).

Lemma 1.1 (see [15]). *Let D be an open subset in $\mathbb{R}^n$ and let $E \subset D$ be compact. Then for any $\tau \in (0, 1)$, $\varepsilon \in (0, 1-\tau)$ and K a compact subset of D_τ there exists a positive constant $c = c(\tau, \varepsilon, K, D)$ such that for any harmonic function f on D we have*

$$\|f\|_K \leq c\|f\|_E^{1-\tau-\varepsilon} \|f\|_D^{\tau+\varepsilon},$$

where

$$D_\tau := \left\{x \in D : \chi_0(D, E, x) < \tau\right\}.$$

Remark 1.2. *It is clear that if $\chi_0(\cdot, E, D) \not\equiv 1$, then E is the uniqueness set for harmonic functions on D. We cannot replace the set D_τ by $\{x \in D : h(x, E, D) \leq \tau\}$, since there exist such compacts E that $h(\cdot, E, D) \not\equiv 1$ and E is not a uniqueness set for harmonic functions on D.*

Using the Zahariuta's extremal functions Hecart proved the following theorem on the analytic continuation of separately harmonic functions.

Theorem 1.1 (see [16]). *Let D and G be domains from the space $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and two H-regular compacts $E \subset D$ and $F \subset G$ are given. Then any separately harmonic function on the set*

$$X = (E \times G) \cup (D \times F)$$

continues harmonically into the domain

$$\hat{X} = \left\{(x,y) \in D \times G : \bar{h}(x,E,D) + \bar{h}(y,F,G) < 1\right\},$$

where

$$\bar{h}(x,E,D) = \lim_{s\to+\infty} h(x,E,D_s)$$

and $D_s \nearrow D$, $s \to +\infty$.

This theorem for the case $n = m = 2$ was previously proved by Zeriahi.

Theorem 1.2 (Zeriahi [12]). *Let* $D \times G$ *be a domain of the space* $\mathbb{R}^2(x) \times \mathbb{R}^2(y)$ *and let* $E \subset D$, $F \subset G$ *be compact sets satisfying the* H*-regularity conditions in the classes of harmonic polynomials. Then any separately harmonic function on the set*

$$X = (E \times G) \cup (D \times F)$$

continues harmonically into the domain

$$\hat{X} = \{(x,y) \in D \times G : \omega^*_{sh}(x,E,D) + \omega^*_{sh}(y,F,G) < 1\}.$$

Here ω^*_{sh} is an ordinary harmonic measure.

2 Main Results

Usually, for continuations of harmonic functions we first pass to holomorphic functions and then use the principles of holomorphic continuations. The following lemma will allow us to make such a transition, and which we will use quite often.

Lemma 2.1 (see [1] and [10]). *Let us consider a space* $\mathbb{R}^n(x)$*, nested in* $\mathbb{C}^n(z) = \mathbb{R}^n(x) + i\cdot\mathbb{R}^n(y)$*, where* $z = (z_1,...,z_n)$*,* $z_j = x_j + i\cdot y_j$*,* $j = 1,...,n$*, and let* D *be some bounded domain of* $\mathbb{R}^n(x)$*. Then there exists a domain* $\widehat{D} \subset \mathbb{C}^n(z)$ *such that* $D \subset \widehat{D}$ *and for any function* $u(x) \in h(D)$*, there exists a function* $f_u(z)$ *holomorphic in* $\widehat{D}$ *such that* $f_u|_D = u$*. Furthermore, for any number* $M > 1$ *there exists a subdomain* $\widehat{D}_M \subset \widehat{D}$*,* $D \subset \widehat{D}_M$*, such that*

$$\|f_u\|_{\widehat{D}_M} \le M\|u\|_D$$

for all $u \in h(D) \cap L^\infty(D)$.

Using Lemma 2.1 and the continuation theorem for separately analytic functions, the following theorem is proved.

Theorem 2.1 (Sadullaev and Imomkulov [10]). *Let* $E \subset D \subset \mathbb{R}^n$ *and* $F \subset G \subset \mathbb{R}^m$ *be compact sets that are nonpluripolar in the sense of subsets of spaces* $\mathbb{C}^n(z) = \mathbb{R}^n(x) + i\cdot\mathbb{R}^n(y)$ *and* $\mathbb{C}^m(z) = \mathbb{R}^m(x) + i\cdot\mathbb{R}^m(y)$*. Then any separately harmonic function* $u(x,y)$ *on the set*

$$X = (E \times G) \cup (D \times F)$$

continues harmonically into the domain

$$\hat{X} = \left\{(x,y) \in D \times G : \omega^*(x,E,\widehat{D}) + \omega^*(y,F,\widehat{G}) < 1\right\}.$$

Here $\omega^*(z,E,\widehat{D})$ and $\omega^*(w,F,\widehat{G})$ are P-measures of the sets E and F with respect to the domains $\widehat{D}$ and $\widehat{G}$, where $E\subset\widehat{D}\subset\mathbb{C}^n$, $F\subset\widehat{G}\subset\mathbb{C}^m$ (see [9]) and

$$\omega^*(z,E,\widehat{D})=\overline{\lim_{z'\to z}}\,\omega(z',E,\widehat{D}),\ \ z\in\widehat{D},$$

where

$$\omega(z,E,\widehat{D})=\sup\left\{u(z):u\in psh\left(\widehat{D}\right),\,u|_E\le 0,\,u|_{\widehat{D}}\le 1\right\}.$$

In particular, an analogue of the Hartogs lemma for separately harmonic functions can be obtained from this theorem.

Lemma 2.2. *Let $u(x,y)$ be a separately harmonic function in the domain*

$$U\times V_r=\{x\in\mathbb{R}^n:|x|<1\}\times\left\{y\in\mathbb{R}^2:|y|<r\right\}\subset\mathbb{R}^n\times\mathbb{R}^2$$

and for each fixed $x^0\in U$ the function $u(x^0,y)$ of variable y continues harmonically into the great circle

$$\left\{y\in\mathbb{R}^2:|y|<R\right\},R>r.$$

Then the function $u(x,y)$ continues harmonically into the domain

$$U\times\left\{y\in\mathbb{R}^2:|y|<R\right\}$$

over the set of variables.

For a more accurate result in this direction, it is necessary to introduce a special measure adapted to harmonic functions. As is known, the P-measure $\omega^*(z,E,D)$ of the compact $E\subset D$ for strongly pseudoconvex domains $D\subset\mathbb{C}^n$ can also be determined using holomorphic functions:

$$\omega(z,E,D)=\sup\left\{\alpha\ln|f(z)|:f\in\mathcal{O}(D),\|f\|_E\le 1,\|f\|_D^\alpha\le e,\alpha>0\right\},$$
$$\omega^*(z,E,D)=\overline{\lim_{\zeta\to z}}\,\omega(\zeta,E,D). \tag{1}$$

The proof of relations (1) follows easily from the Bremermann theorem on the approximation of plurisubharmonic functions by plurisubharmonic Hartogs functions (see [9]).

As we noted above in Lemma 2.1, any harmonic function $u\in h(D)$ continues holomorphically into a fixed domain $\widehat{D}$ of the space $\mathbb{C}^n, D\subset\mathbb{R}^n\subset\mathbb{C}^n,\widehat{D}\supset D$ and there exists $f\in\mathcal{O}(\widehat{D})$ such that $f|_D\equiv u$. Moreover, by Lemma 2.1, the domain $\widehat{D}$ can be chosen such that

$$\|f\|_{\widehat{D}}\le M\|u\|_D,$$

where M is a constant independent of u.

From this we see that the quantity

$$\gamma^*(z,E,\widehat{D})=$$
$$=\overline{\lim_{w\to z}}\sup\left\{\alpha\ln|f(w)|:f\in\mathcal{O}(\widehat{D}),\,f|_D\in h(D),\|f\|_E\le 1,\|f\|_{\widehat{D}}^\alpha\le e,\alpha>0\right\},$$

which we will call h-measure, differs from the P-measure $\omega^*(z,E,\widehat{D})$ only by the fact that in its definition includes an additional harmonicity condition $f|_D$. Nevertheless, the sets of the zero h-measure are finer than the sets of the zero P-measure.

Example 1. *The a set* $S = \{x \in \mathbb{R}^n : |x| = \frac{1}{2}\} \subset B(0,1)$ *the measure is* $\gamma^*(x, S, B) \not\equiv 1$*, but* $\omega^*(z, S, \widehat{B}) \equiv 1$.

The h-measure $\gamma^*(x, E, D)$ is more suitable for accurate estimates in contrast to the quantity $h(x, E, D)$ which appears in the works of Hecart [15] and [16].

The following theorem holds.

Theorem 2.2. *Let the function* $u(x, y)$ *be separately harmonic in the domain*

$$D \times V_r = D \times \left\{y \in \mathbb{R}^2 : |y| < r, r > 1\right\} \subset \mathbb{R}^n \times \mathbb{R}^2$$

and for each $x^0 \in D$ *the function* $u(x^0, y)$ *of variable* y *continues harmonically into the great circle*

$$\left\{y \in \mathbb{R}^2 : |y| < R(x^0), R(x^0) > r\right\}.$$

Then the function $u(x, y)$ *continues harmonically into the domain*

$$\left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^2 : |y| < R_*(x), x \in D\right\}$$

over a set of variables, where the function $\ln \frac{1}{R_*(x)}$ *is the trace of some plurisubharmonic function from the class* $\mathcal{H}(\widehat{D})$.

First, we define the following class of plurisubharmonic functions: $\mathcal{H}(\widehat{D})$ is the minimal class of plurisubharmonic functions that contains all functions of the form $\alpha \ln |f(z)|$, where $f(z) \in \mathcal{O}(\widehat{D}) \cap h(D), \alpha > 0$, and is closed with respect to the operation "upper regularisation", i.e., for any family $u_\lambda(z) \in \mathcal{H}(\widehat{D})$, $\lambda \in \Lambda$ of locally upper bounded functions, the function

$$u(z) = \overline{\lim_{\zeta \to z}} \sup \{u_\lambda(\zeta) : \lambda \in \Lambda\}$$

also belongs to the class $\mathcal{H}(\widehat{D})$.

It is not difficult to verify that $\mathcal{H}(\widehat{D}) \subset psh(\widehat{D})$ and $\mathcal{H}(\widehat{D}) \neq psh(\widehat{D})$.

Let

$$u(z) = \overline{\lim_{j \to +\infty}} u_j(z),$$

where $u_j(z) \in psh(\widehat{D})$ is a sequence of locally upper bounded plurisubharmonic functions. Then the regularisation of the limit function $u(z)$

$$u^*(z) = \overline{\lim_{\zeta \to z}} u(\zeta)$$

is a plurisubharmonic function in the domain of functions $u_j(z)$. Sadullaev (see [9]) proved that the set $\{z : u(z) < u^*(z)\}$ is pluripolar, and conversely any pluripolar set is represented as a set of type $\{z : u(z) < u^*(z)\}$. Indeed, let $u(z)$ is a plurisubharmonic function, where $u(z) \not\equiv -\infty$ and $u(z)|_E = -\infty$. Consider the sequence $u_j(z) = \frac{1}{j} u(z)$. It is clear that

$$u(z) = \overline{\lim_{j \to \infty}} u_j(z) = 0$$

almost everywhere and

$$u(z) = \overline{\lim_{j\to\infty}} u_j(z) = -\infty$$

for all $z \in E$. It follows that

$$E \subset \{z : u(z) < u^*(z)\}.$$

So the inclusion $E \subset \{z : u(z) < u^*(z)\}$ can be considered as an equivalent definition of a pluripolar set.

If a sequence of functions $u_j(z) \in \mathcal{H}(\widehat{D})$ is locally uniformly upper bounded, then the regularisation of the upper limit $u^*(z)$, by definition, also belongs to the class $\mathcal{H}(\widehat{D})$. In this case, we call the pluripolar set $E \subset \{x \in D \subset \mathbb{R}^n : u(x) < u^*(x)\}$ an h-pluripolar set.

It is clear that the pluripolar sets E_1 and E_2, where $u|_{E_1} = -\infty$, $u(z) \in \mathcal{H}(\widehat{D})$, $u(z) \not\equiv -\infty$ and $\gamma^*(z, E_2, \widehat{D}) \equiv 1$, are also h-pluripolar.

Proof theorem 2.2. Let the function $u(x, y)$ be separately harmonic in the domain $D \times V_r$. Consider the function $u(\cdot, y)$ as a function of the complex variable $w = y_1 + iy_2$ ($y = (y_1, y_2)$), i.e., as a function $u(\cdot, w)$, $w \in V_r \subset \mathbb{C} \simeq \mathbb{R}^2$, $r > 1$. Since the function $u(x, w)$ is harmonic with respect to w in the neighbourhood of the unit circle $V_1 \subset V_r$, therefore there is a decomposition into a Hartogs series:

$$u(x, w) = u(x, \rho e^{i\varphi}) = \sum_{k=-\infty}^{+\infty} c_k(x) \rho^{|k|} e^{ik\varphi}, \tag{2}$$

where

$$c_k(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x, e^{i\varphi}) e^{-ik\varphi} d\varphi.$$

It is clear that the coefficients $c_k(x) \in h(D)$ and they are locally uniformly bounded, i.e., for any $K \Subset D$ there exists a number $N > 0$, such that

$$|c_k(x)| \leq N, \forall x \in K, \forall k \in \mathbb{Z}.$$

From here, by virtue of Lemma 2.1, there exists a domain $\widehat{D} \subset \mathbb{C}^n$, such that $D \subset \widehat{D}$ and all coefficients holomorphically continue into $\widehat{D}$, i.e., there exist holomorphic functions $\hat{c}_k(z) \in \mathcal{O}(\widehat{D}) \cap h(D)$, such that $\hat{c}_k(z)|_D \equiv c_k(x)$. Moreover, the sequence of functions $\hat{c}_k(z)$ is also locally uniformly bounded.

Hence, the sequence of plurisubharmonic functions

$$\frac{1}{|k|} \ln |\hat{c}_k(z)| \in \mathcal{H}\left(\widehat{D}\right)$$

is locally uniformly upper bounded and the function

$$v^*(z) = \overline{\lim_{\zeta\to z}}\, \overline{\lim_{k\to+\infty}} \frac{1}{|k|} \ln |\hat{c}_k(z)|$$

also belongs to the $\mathcal{H}(\widehat{D})$ class. In addition, according to the conditions of the theorem, for each fixed $x \in D$ the next relation holds

$$\lim_{|k|\to+\infty} \sqrt[|k|]{|c_k(x)|} = \frac{1}{R(x)},$$

i.e., $v^*(z)|_D = \ln \frac{1}{R(x)}$ outside of some h-pluripolar set from the domain D. More precisely

$$v^*(x) \equiv \ln \frac{1}{R_*(x)}$$

in the domain D, since the domain $D \subset \mathbb{R}^n$ is not pluri-thin at any point $x \in D$. According to the estimate from Lemma 2.1, the following series

$$\hat{u}(z,w) = \sum_{k=-\infty}^{+\infty} \hat{c}_k(z)\rho^{|k|}e^{ik\varphi} \left(w = \rho e^{i\varphi},\, \hat{u}(z,w)|_{D\times V_r} = u(x,w)\right) \tag{3}$$

converges locally uniformly in the domain $\widehat{D}\times V_r$ and for each fixed $z \in \widehat{D}$ continues on the variable w to a harmonic function in $\{w : |w| < e^{-v^*(z)}\}$, where $e^{-v^*(x)} = R_*(x)$, $x \in D$.

Let us take now any point $x^0 \in D$ and number $\sigma : r < \sigma < R_*(x^0)$. According to the lower semi-continuity of the function $R_*(x)$ there exists some δ-neighborhood $U_\delta = \{x : |x - x^0| < \delta\}$ of a point x^0 such that $R(x) > \sigma$ for any $x \in U_\delta$. Hence, according to Lemma 2.2, we obtain that the function $u(x,w) = u(x,y)$ continues harmonically into the domain $U_\delta \times \{y : |y| < \sigma\}$ over the set of variables. Finally, from the arbitrariness of the point $x^0 \in D$ and the number $\sigma : r < \sigma < R_*(x^0)$ we obtain that the function $u(x,y)$ continues harmonically into the domain $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^2 : |y| < R_*(x), x \in D\}$ and by definition the function $\ln \frac{1}{R_*(x)}$ is the trace of the plurisubharmonic function $v^*(z) \in \mathcal{H}(\widehat{D})$. □

Theorem 2.3. *Let $u(x,y)$ be separately harmonic function in the domain*

$$D \times V_r = D \times \left\{y \in \mathbb{R}^2 : |y| < r, r > 1\right\} \subset \mathbb{R}^n \times \mathbb{R}^2$$

and for each fixed $x^0 \in E \subset D$, where the set E is not h-pluripolar, the function $u(x^0,y)$ of variable y continues harmonically over the all plane $\mathbb{R}^2$. Then the function $u(x,y)$ continues harmonically into the domain $D \times \mathbb{R}^2$ over the set of variables.

Proof. Assuming, as in the proof of Theorem 2.2, for each fixed $x \in D$ we decompose the function

$$u(x,y) = u(x,w), w = y_1 + iy_2 \,(y = (y_1,y_2))$$

into the following Hartogs series

$$u(x,w) = u(x,\rho e^{i\varphi}) = \sum_{k=-\infty}^{+\infty} c_k(x)\rho^{|k|}e^{ik\varphi}, \tag{4}$$

where

$$c_k(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(x, e^{i\varphi}) e^{-ik\varphi} d\varphi.$$

Clearly, the coefficients $c_k(x) \in h(D)$ and they are locally uniformly bounded. Hence, by virtue of Lemma 2.1, there exists a domain $\widehat{D} \subset \mathbb{C}^n$ such that $D \subset \widehat{D}$ and all coefficients holomorphically continue into $\widehat{D}$ i.e., there exist holomorphic functions with $\hat{c}_k(z) \in \mathcal{O}(\widehat{D}) \cap h(D)$ such that $\hat{c}_k(z)|_D \equiv c_k(x)$. Moreover, the sequence of holomorphic functions $\hat{c}_k(z)$ is also locally uniformly bounded. Hence, the sequence of plurisubharmonic functions

$$\frac{1}{|k|} \ln |\hat{c}_k(z)| \in \mathcal{H}\left(\widehat{D}\right)$$

is locally uniformly upper bounded and the function

$$\upsilon^*(z) = \overline{\lim_{\zeta \to z}}\, \overline{\lim_{k \to +\infty}} \frac{1}{|k|} \ln |\hat{c}_k(z)|$$

also belongs to the class $\mathcal{H}\left(\widehat{D}\right)$. Moreover, by the conditions of theorem $\upsilon^*(z)|_E = -\infty$ and E is not h-pluripolar, therefore

$$\upsilon^*(z) \equiv -\infty$$

in the domain $\widehat{D}$, i.e. function

$$\hat{u}(z, w) = \sum_{k=-\infty}^{+\infty} \hat{c}_k(z) \rho^{|k|} e^{ik\varphi} \left(w = \rho e^{i\varphi},\ \ \hat{u}(z, w)|_{D \times V_r} = u(x, w)\right) \tag{5}$$

for each fixed $z \in \widehat{D}$ continues on the variable w to function that is harmonic in the all plane $\mathbb{C}$. Finally, applying the Hartogs lemma (see [18], page 331, theorem 8) to the sequence of functions

$$\frac{1}{|k|} \ln |\hat{c}_k(z)|$$

we obtain that the series (5) is locally uniformly convergent in the domain $\widehat{D} \times \mathbb{C}$, i.e., the series (4) is locally uniformly convergent in the domain $D \times \mathbb{C} \approx D \times \mathbb{R}^2$ and its sum is a harmonic continuation of the function $u(x, y)$. □

References

[1] Avanissian V. Cellule d'Harmonicite et Prolongement Analitique Complexe. Travaux en cours, Hermann, Paris (1985).

[2] Gonchar A.A. On analytic continuation from the "edge of the wedge". Annales Academire Scientiarum Fennicre Series A. I. Mathematica. Vol. 10, pp. 221-125 (1985).

[3] Zakharyuta V.P. Separately analytic functions, generalized Hartogs theorems and holomorphy shells. Mat. Sat. Vol. 101, Issue 1, pp. 57-76 (1976).

[4] Zahariuta V.P. Spaces of harmonic functions, in: Functional Analysis, Lecture Notes in Pure and Applied Math. Vol. New York, 150, pp. 497-522 (1994).

[5] Sičiak J. Separately analytic functions and envelopes of holomorphy of some lowerdimensional subsets of $\mathbb{C}^n$. Ann. Pol. Math. Vol. 22, Issue 1, pp. 145-171 (1969).

[6] Sičiak J. Asymptotic behaviour of harmonic polynomials bounded on a compact set. Ann. Pol. Math. Vol. 20, 267-278 (1968).

[7] Sičiak J. Bernstein-Walsh type theorems for pluriharmonic functions, in: Potential Theory-Proceedings of the International Conference, Kouty, 13-20 August 1994, J. Král et al. (eds.), Walter de Gruyter, Berlin-New York, 147-166 (1996).

[8] Sičiak J. Bernstein-Walsh Theorems for Elliptic Operators. Jagiellonian University (1997) (preprint).

[9] Sadullaev A.S. Plurisubharmonic functions. Results of science and technology. Modern problems of mathematics. Fundamental directions. M.: VINITI, Vol. 8, 65-111 (1985).

[10] Sadullaev A.S., Imomkulov S.A. Continuation of holomorphic and pluriharmonic functions with subtle singularities on parallel sections. Proceedings of the Mathematical Institute named after V.A. Steklova Vol. 53, 158-174 (2006).

[11] Nguyen T.V., Zeriahi A. Une extension du theoreme de Hartogs sur les fonctions separement analytiques. Analyse Complexe Multivariables, Recents Developments, A. Meril (ed), Editel, Rene, pp. 183-194 (1991).

[12] Zeriahi A. Bases communes dans certains espaces de functions harmoniques et functions sur certains ensemblesde $\mathbb{C}^n$. Ann. Fac. Sci. Toulouse Nouvelle. Vol. 4, Issue 5, pp. 75-102 (1982).

[13] Pflug P., Nguyen V.A. A boundary cross theorem for separately holomorphic functions. Ann. Polon. Math. Vol. 84, pp. 237-271 (2004).

[14] Lelong P. Fonctions plurisousharmoniques et fonctions analytiques de variables reelles. Ann. Inst. Fourier. Vol. 11, pp. 515-562 (1961).

[15] Hecart J. On Zahariutas extremal functions for harmonic functions. Vietnam. J. Math. Vol. 27, Issue 1, pp. 53-59 (1999).

[16] Hecart J. Harmonicity domains far Separately harmonic functions. Potential. Anal. Vol. 13, pp. 115-126 (2000).

[17] Van N.T., Djebbar B. Propriétés asymptotiques d'une suite orthonormale de polynômes harmoniques. Bull. Soc. Math. Issue 113, pp. 239-251 (1989).

[18] Shabat B.V. Introduction to complex analysis. Part I. Functions of one variable. Moscow "Science" (1985).