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TRANSLATION-INVARIANT GIBBS MEASURES FOR POTTS MODEL WITH COMPETING INTERACTIONS WITH A COUNTABLE SET OF SPIN VALUES ON CAYLEY TREE

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Abstract

In the present paper we consider of an infinite system of functional equations for the Potts model with competing interactions and countable spin values $\Phi = \{0, 1, ..., \}$ on a Cayley tree of order k. We study translation-invariant Gibbs measures that gives the description of the solutions of some infinite system of equations. For any $k \geq 1$ and any fixed probability measure ν we show that the set of translation-invariant splitting Gibbs measures contains one and two points for odd k and even k, respectively, independently on parameters of the Potts model with a countable set of spin values on a Cayley tree.

Keywords: *Gibbs measure, Cayley tree, Potts model, Hamiltonian, nearest-neighbor, countable spin values, configuration, functional equation, translation-invariant solution, probability measure, competing interaction.*

Mathematics Subject Classification (2020): *82B26, 60K35.*

Introduction

The Gibbs measure is a probability measure which has been an important object in many problems of probability theory and statistical mechanics. It is the measure associated with Hamiltonian of a physical system (model) [1].

The method used for the description of Gibbs measures on Cayley trees is the method of Markov random field theory and recurrent equations of this theory, but modern theory of Gibbs measures on trees uses new tools such as group theory, contour methods on trees, non-linear analysis.

The most studied model of statistical mechanics is the Ising model, there are about 1700 papers devoted to the problems related to Ising model. In particular, this model plays a very special role in statistical mechanics and gives the simplest nontrivial example of a system undergoing phase transitions [3], [5], [17], [18], [19].

The Potts model was introduced as a generalization of the Ising model. The idea came from the representation of the Ising model as interacting spins which can be either parallel or antiparallel. At present the Potts model encompasses a number of problems in statistical physics and lattice theory (see, e.g., [2]). It has been a subject of increasing research interest in recent years.

In [5] a countable state space Markov random fields and Markov chains on trees were constructed and using of entrance laws for specifications Zachary extended and generalized results of [3], [6].

In [8] the ferromagnetic Potts model with three states on a second order Cayley tree was studied and there was shown the existence of a critical temperature T_c such that for $T < T_c$ there are three translation-invariant Gibbs measures and uncountable Gibbs measures, which are not translation-invariant. In [12], the results of [8] were generalized for the Potts model with a finite number of states on a Cayley tree of arbitrary (finite) order. It was proved in [9] that the translation-invariant Gibbs measure of the antiferromagnetic Potts model with an external field is unique on the Cayley tree. [10] was devoted to the Potts model with countable many states and a non-zero external field on the Cayley tree. It was proved that this model has a unique translation-invariant Gibbs measure. In [7] the Potts model with a countable set of spin values on Z^d was studied. Moreover, [23] is devoted to discussing many applications of the Potts model to real world situations, such as biology, physics, and some examples of alloy behavior, cell sorting, financial engineering, image segmentation, medicine, sociology and given a systematic review of the theory of Gibbs measures of Potts model on Cayley trees.

In [22] all translation-invariant splitting Gibbs measures (TISGMs) were found on the Cayley tree for the Potts model, in particular, it was shown that at sufficiently low temperatures their number is equal to $2^q - 1$. It was proved that there were $[\frac{q}{2}]$ critical temperatures and the exact amount of TISGMs for each temperature was given and in [24] the regions of non-extremality of these measures were found.

In [11] it was considered a nearest-neighbor Potts model, with countable spin values $\Phi = \{0, 1, ...,\}$ and non-zero external field, on a Cayley tree of order k (with $k+1$ neighbors). Also it was given full description of the class of probabilistic measures ν on Φ and in particular it was described the Poisson measures which are Gibbsian.

Ground states for Potts model with a countable set of spin values on a Cayley tree were considered in [14], [15], [16] and [21].

The chapter 8 in [4] was devoted to a nearest-neighbor Potts model with countable spin values $0, 1, ...,$ and non-zero external field on a Cayley tree of order k and studied translation-invariant Gibbs measures which depend on k and a probability measure ν. In [13] an infinite system of functional equations for the Potts model with competing interactions of radius $r = 2$ and countable spin values on a Cayley tree of order two were given and the exact value of the exponential solutions were described such that $u_i = a^i$ for some $a \in (0, 1)$ and the corresponding measure ν.

In this paper, we describe of an infinite system of functional equations for the Potts model with competing interactions on a Cayley tree and give full analysis of the system of equations (12) below.

1 Main definitions and known facts

The Cayley tree (Bethe lattice) Γ^k of order $k \geq 1$ infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the of edges. Two vertices x and y are called *nearest-neighbors* if there exists an edge $l \in L$ connecting them and we denote $l =< x, y >$.

A collection of nearest neighbor pairs $< x, x_1 >, < x_1, x_2 >, ..., < x_{d-1}, y >$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x and y.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n,$$

the set of direct successors of x.

The vertices x and y are called *next-nearest-neighbor* (NNN) which is denoted by $> x, y <$, if there exists a vertex $z \in V$ such that x, z and y, z are nearest-neighbor. We consider NNN $> x, y <$, for which there exist n such that $x \in W_n$ and $y \in W_{n+2}$, this kind of NNN is considered with the three states Potts model (see [20]).

It is well-known that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k + 1$ cyclic groups of second order with generators $a_1, a_2, ..., a_{k+1}$ (see [4], p.3).

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k :< x, y >\in L\}$. The set $S_1(x) \backslash S(x)$ is a singleton.

We consider a Potts model with competing nearest-neighbor and prolonged next-nearest-neighbor interactions on a Cayley tree where the spin takes values in the set $\Phi := 0, 1, 2, ...$ A configuration σ on V is then defined as a function $x \in V \longmapsto \sigma(x) \in \Phi$; the set of all configurations is Φ^V.

Let ${G_k}^*$ be a subgroup of index $r \geq 1$. We consider the right coset $G_k \setminus {G_k}^* = \{H_1, H_2, ..., H_r\}$.

Definition 1. *A configuration $\sigma(x)$ is said to be ${G_k}^* -periodic$ if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k-periodic configuration is said to be translation-invariant.*

The Hamiltonian of the Potts model with competing interactions has the form

$$H(\sigma) = -J \sum_{\substack{<x,y> \\ x,y \in V}} \delta_{\sigma(x)\sigma(y)} - J_1 \sum_{\substack{>x,y< \\ x,y \in V}} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J, J_1 \in R$ are coupling constants and δ is the Kroneker's symbol:

$$\delta_{uv} = \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}$$

For $A \subset V$ denote by Φ^A the configuration space on A. Let $h : x \longmapsto h_x = (h_{0,x}, h_{1,x}, ...) \in R^\infty$ be a real sequence-valued function of $x \in V \setminus \{x^0\}$.

Fix a probability measure $\nu = \{\nu(i) > 0, i \in \Phi\}$.

Given $n = 1, 2, ...$, consider the probability distribution μ_n on Φ^{V_n} defined by

$$\mu^{(n)}(\sigma_n)=Z_n^{-1}\exp\left(-\beta H(\sigma_n)+\sum_{x\in W_n}h_{\sigma(x),x}\right)\prod_{x\in V_n}\nu(\sigma(x)). \quad (2)$$

Here, $\sigma_n : x\in V_n\longmapsto\sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n=\sum_{\tilde{\sigma}_n\in\Phi^{V_n}}\exp\left(-\beta H(\tilde{\sigma}_n)+\sum_{x\in W_n}h_{\tilde{\sigma}(x),x}\right)\prod_{x\in V_n}\nu(\tilde{\sigma}(x)).$$

Remark 1. *Note that Z_n is the finite, since ν is a probability measure and* $\exp(-\beta H(\tilde{\sigma}_n)+\sum\limits_{x\in W_n}h_{\tilde{\sigma}(x),x})$ *is bounded on Φ^{V_n}.*

As usual, the probability distributions $\mu^{(n)}$ are compatible if for any $n\geq 1$ and $\sigma_{n-1}\in\Phi^{V_{n-1}}$:

$$\sum_{\omega_n\in\Phi^{W_n}}\mu^{(n)}(\sigma_{n-1}\vee\omega_n)=\mu^{(n-1)}(\sigma_{n-1}). \quad (3)$$

Here $\sigma_{n-1}\vee\omega_n\in\Phi^{V_n}$ is the concatenation of σ_{n-1} and ω_n.

The following theorem describes conditions on h_x guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

2 Functional Equations

Theorem 1. *Probability distributions $\mu^{(n)}(\sigma_n)$, $n=1,2,...$, in (2), for a Cayley tree of order two, are compatible iff for any $x\in V\backslash\{x^0\}$ the following equation holds:*

$$h^*_{i,x}=F_i(h^*_y,h^*_z,\beta,J),\quad i=1,2,..., \quad (4)$$

where $S(x)=\{y,z\}$, $\quad h^*_x=(h_{1,x}-h_{0,x}+\ln\frac{\nu(1)}{\nu(0)},h_{2,x}-h_{0,x}+\ln\frac{\nu(2)}{\nu(0)},...)$ *and*

$$F_i(h^*_y,h^*_z,\beta,J)=\ln\frac{1+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}\exp\left\{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}+h^*_{p,y}+h^*_{q,z}\right\}}{1+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}\exp\left\{\beta J(\delta_{0p}+\delta_{0q})+J_1\beta\delta_{pq}+h^*_{p,y}+h^*_{q,z}\right\}}.$$

Proof. Necessity Assume that (3) holds; we will prove (4). Substituting (2) in (3), obtain that for any configurations $\sigma_{n-1} : x\in V_{n-1}\longmapsto\sigma_{n-1}(x)\in\Phi$:

$$\frac{1}{Z_n}\sum_{\sigma^{(n)}\in\Phi^{W_n}}\exp\left\{-\beta H_n(\sigma_n)+\sum_{x\in W_n}h_{\sigma(x),x}\right\}\times\prod_{y\in V_{n-1}}\nu(\sigma(y))$$

$$= \frac{1}{Z_{n-1}} \exp \left\{ -\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \right\}.$$

$$\sum_{\sigma^{(n)} \in \Phi^{W_n}} \exp \left\{ -\beta H_{n-1}(\sigma_{n-1}) + J\beta \sum_{\substack{x \in W_{n-1} \\ y,z \in S(x)}} \left(\delta_{\sigma(x)\sigma(y)} + \delta_{\sigma(x)\sigma(z)} \right) + J_1 \beta \sum_{\substack{x \in W_{n-1} \\ y,z \in S(x)}} \delta_{\sigma(y)\sigma(z)} + \sum_{x \in W_n} h_{\sigma(x),x} \right\}$$

$$\times \prod_{y \in W_n} \nu(\sigma(y)) = \frac{Z_n}{Z_{n-1}} \exp \left\{ -\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} \exp h_{\sigma_{n-1}(x),x} \right\}.$$

Consequently, for any $i \in \Phi$,

$$\frac{\exp\left\{h_{0,y} + h_{0,z} + 2\ln \nu(0)\right\} + \sum\limits_{\substack{p,q=0 \\ p+q \neq 0}}^{\infty} \exp\left\{J\beta(\delta_{ip} + \delta_{iq}) + J_1\beta\delta_{pq} + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\right\}}{\exp\left\{h_{0,y} + h_{0,z} + 2\ln \nu(0)\right\} + \sum\limits_{\substack{p,q=0 \\ p+q \neq 0}}^{\infty} \exp\left\{J\beta(\delta_{0p} + \delta_{0q}) + J_1\beta\delta_{pq} + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\right\}}$$

$$= \exp\left\{h_{i,x} - h_{0,x}\right\},$$

such that:

$$h^*_{i,x} = \ln \frac{1 + \sum\limits_{\substack{p,q=0 \\ p+q \neq 0}}^{\infty} \exp\left\{J\beta(\delta_{ip} + \delta_{iq}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\right\}}{1 + \sum\limits_{\substack{p,q=0 \\ p+q \neq 0}}^{\infty} \exp\left\{J\beta(\delta_{0p} + \delta_{0q}) + J_1\beta\delta_{pq} + h^*_{p,y} + h^*_{q,z}\right\}},$$

where:

$$h^*_{i,x} = h_{i,x} - h_{0,x} + \ln \frac{\nu(i)}{\nu(0)}.$$

Sufficiency. Let (4) is satisfied we will prove (3).

$$\sum_{p,q=0}^{\infty} \exp\left\{J\beta(\delta_{ip} + \delta_{iq}) + J_1\beta\delta_{pq} + h_{p,y} + h_{q,z} + \ln\nu(p) + \ln\nu(q)\right\} = a(x) \exp\left\{h_{i,x}\right\}, \quad (5)$$

here $i = 0, 1, ...$

We have

$$\text{LHS of (3)} = \frac{1}{Z_n} \exp\left\{ -\beta H_{n-1}(\sigma_{n-1})\right\} \prod_{x \in W_{n-1}} \nu(\sigma(x)) \times$$

$$\sum_{\substack{x\in W_{n-1}\\ y,z\in S(x)}} \exp\left\{J\beta\big(\delta_{\sigma(x)\sigma(y)}+\delta_{\sigma(x)\sigma(z)}\big)+J_1\beta\delta_{\sigma(y)\sigma(z)}+h_{\sigma(y),y}+h_{\sigma(z),z}\right\}. \tag{6}$$

Substituting (5) into (6) and denoting $A_n=\prod_{x\in W_{n-1}} a(x)$, we get:

$$\text{RHS of (3.3)} = \frac{A_{n_1}}{Z_n}\exp\left\{-\beta H_{n-1}(\sigma_{n-1})\right\}\prod_{x\in W_{n-1}}\nu(\sigma(x)). \tag{7}$$

Since $\mu^{(n)}, n\geq 1$ is a probability, we should have:

$$\sum_{\sigma_{n-1}}\sum_{\sigma}^{(n)}\mu^{(n)}\big(\sigma_{n-1},\sigma^{(n-1)}\big)=1.$$

Hence, from (7) we obtain $Z_{n-1}A_{n-1}=Z_n$, and (3) holds. □

3 Translation-invariant solutions

Let, $h_x=h=(h_1,h_2,...)$ for any $x\in V$. We rewrite (4) as the following form:

$$h_i=\ln\frac{\nu(i)}{\nu(0)}+k\ln\frac{1+\sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty}\exp\left\{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}+h_p^*+h_q^*\right\}}{1+\sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty}\exp\left\{\beta J(\delta_{0p}+\delta_{0q})+J_1\beta\delta_{pq}+h_p^*+h_q^*\right\}}. \tag{8}$$

From following determine $u_i=\exp(h_i), i=1,2,...$ we have

$$u_i=\frac{\nu(i)}{\nu(0)}\cdot\left(\frac{1+\sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty}\exp\left\{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}\right\}u_pu_q}{1+\sum\limits_{\substack{p,q=0\\ p+q\neq 0}}^{\infty}\exp\left\{\beta J(\delta_{0p}+\delta_{0q})+J_1\beta\delta_{pq}\right\}u_pu_q}\right)^k,\ i\in N. \tag{9}$$

Using the following expressions:

$$e^{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}}u_pu_q=\theta^2\theta_1u_i^2 \quad \text{when} \quad p=q=i;$$

$$\sum_{p\neq i,q=i}^{\infty}e^{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}}u_pu_q=\theta u_i\sum_{\substack{p>0\\ p\neq i}}^{\infty}u_p;$$

$$\sum_{q\neq i,p=i}^{\infty}e^{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}}u_pu_q=\theta u_i\sum_{\substack{q>0\\ q\neq i}}^{\infty}u_q;$$

$$\sum_{p\neq q\neq i}^{\infty} e^{\beta J(\delta_{ip}+\delta_{iq})+J_1\beta\delta_{pq}} u_p u_q = \sum_{\substack{p>0\\p\neq i}}^{\infty}(u_p \sum_{\substack{q>0\\q\neq i}}^{\infty} \theta_1^{pq} u_q);$$

we rewrite (9) as

$$u_i = \frac{\nu(i)}{\nu(0)} \cdot \left(\frac{1+\theta^2\theta_1 u_i^2 + \theta u_i \sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p + \theta u_i \sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q + \sum\limits_{\substack{p>0\\p\neq i}}^{\infty}(u_p \sum\limits_{\substack{q>0\\q\neq i}}^{\infty} \theta_1^{\delta_{pq}} u_q)}{1+\theta^2\theta_1 u_0^2 + \theta u_0 \sum\limits_{p>0}^{\infty} u_p + \theta u_0 \sum\limits_{q>0}^{\infty} u_q + \sum\limits_{p>0}^{\infty}(u_p \sum\limits_{q>0}^{\infty} \theta_1^{\delta_{pq}} u_q)} \right)^k \tag{10}$$

where $\theta = \exp(J\beta), \theta_1 = \exp(J_1\beta)$.

$$\sum_{\substack{p>0\\p\neq i}}^{\infty}(u_p \sum_{\substack{q>0\\q\neq i}}^{\infty} \theta_1^{\delta_{pq}} u_q) = \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p \left(\{\text{in case } p=q\}\theta_1 u_p + \sum_{\substack{q>0\\q\neq i, p\neq q}}^{\infty} u_q \right) =$$

$$= \theta_1 \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p^2 + \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p \cdot \sum_{\substack{q>0\\q\neq i, p\neq q}}^{\infty} u_q \tag{11}$$

and $u_0 = 1$. From (11) we rewrite

$$\sum_{\substack{p>0\\p\neq i}}^{\infty} u_p \cdot \sum_{\substack{q>0\\q\neq i, p\neq q}}^{\infty} u_q = \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p \cdot \left(\sum_{\substack{q>0\\q\neq i}}^{\infty} u_q - u_p \right) = \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p \cdot \sum_{\substack{q>0\\q\neq i}}^{\infty} u_q - \sum_{\substack{p>0\\p\neq i}}^{\infty} u_p^2.$$

and we get

$$u_i = \frac{\nu(i)}{\nu(0)} \cdot \left(\frac{1+\theta^2\theta_1 u_i^2 + \theta u_i \left(\sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p + \sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q \right) + (\theta_1 - 1)\sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p^2 + \sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p \cdot \sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q}{1+\theta^2\theta_1 + \theta \left(\sum\limits_{p>0}^{\infty} u_p + \sum\limits_{q>0}^{\infty} u_q \right) + (\theta_1 - 1)\sum\limits_{p>0}^{\infty} u_p^2 + \sum\limits_{p>0}^{\infty} u_p \cdot \sum\limits_{q>0}^{\infty} u_q} \right)^k \tag{12}$$

or

$$u_i = \frac{\nu(i)}{\nu(0)} \cdot \left(1 + \frac{\theta^2\theta_1(u_i^2 - 1) + \theta \left(\sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p + \sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q \right)(u_i - 1)}{1+\theta^2\theta_1 + \theta \left(\sum\limits_{p>0}^{\infty} u_p + \sum\limits_{q>0}^{\infty} u_q \right) + (\theta_1 - 1)\sum\limits_{p>0}^{\infty} u_p^2 + \sum\limits_{p>0}^{\infty} u_p \cdot \sum\limits_{q>0}^{\infty} u_q} \right)^k.$$

Let $\sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p = \infty$ and $\sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q = \infty$. From the last equation we get

$$u_i = \frac{\nu(i)}{\nu(0)}, i = 1, 2, ... \tag{13}$$

Since $\sum\limits_{j=0}^{\infty} \nu(j) = 1$ by (13) we have

$$\sum_{j=1}^{\infty} u_j = \frac{1-\nu(0)}{\nu(0)} < +\infty.$$

Thus there is no solution of (12) with $\sum\limits_{\substack{p>0\\p\neq i}}^{\infty} u_p = \infty$ and $\sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q = \infty$.

Let $p > q$ and $\sum\limits_{p>0}^{\infty} u_p = S < +\infty$, $\sum\limits_{\substack{q>0\\q\neq i}}^{\infty} u_q = A + S < +\infty$, $\sum\limits_{p>0}^{\infty} u_p^2 = B < +\infty$, where S,A and B are some fixed positive numbers. It is easy to see that $A = u_q + u_{q+1} + ... + u_{p-1}$.

In this case we obtain from (12)

$$u_i = \frac{\nu(i)}{\nu(0)} \cdot \left(\frac{1 + \theta^2\theta_1 u_i^2 + \theta u_i(A + 2S - 2u_i) + (\theta_1 - 1)(B - u_i^2) + (S - u_i)(A + S - u_i)}{1 + \theta^2\theta_1 + \theta(A + 2S) + (\theta_1 - 1)B + S(A + S)} \right)^k . \tag{14}$$

Denote $\eta_i = \frac{\nu(0)}{\nu(i)}$ and $B_i = \eta_i \left(1 + \theta^2\theta_1 + \theta(A + 2S) + (\theta_1 - 1)B + S(A + S)\right)^k$. Then from (14) we obtain

$$B_i u_i = \left((2 - 2\theta - \theta_1)u_i^2 + (\theta^2\theta_1 + (A + 2S)(\theta - 1))u_i + 1 + (\theta_1 - 1)B + AS + S^2\right)^k . \tag{15}$$

Assuming $\theta_1 \geq 1$, we conclude $B_i > 0$. We will consider cases $k = 1$ and $k = 2$ separately.

Let $k = 1$. Then from (15)

$$B_i u_i = (2 - 2\theta - \theta_1)u_i^2 + (\theta^2\theta_1 + (A + 2S)(\theta - 1))u_i + 1 + (\theta_1 - 1)B + AS + S^2. \tag{16}$$

Case $\theta = \theta_1 = 1$. From (16) we have the following equation

$$B_i u_i = -u_i^2 + u_i + 1 + AS + S^2$$

and one positive solution:

$$u_i = \frac{-(B_i - 1) + \sqrt{(B_i - 1)^2 + 4(1 + AS + S^2)}}{2}$$

Case $\theta = 1$ **and** $\theta_1 > 1$. In this case from (16) we get

$$B_i u_i = -\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1 - 1)B,$$

accordingly

$$u_i = \frac{-(B_i-\theta_1)+\sqrt{(B_i-\theta_1)^2+4\theta_1(1+AS+S^2+(\theta_1-1)B)}}{2\theta_1} > 0.$$

Case $\theta > 1$ **and** $\theta_1 \geq 1$. Denote

$$a = 2-2\theta-\theta_1, \quad b = \theta^2\theta_1 + (A+2S)(\theta-1), \quad c = 1+(\theta_1-1)B + AS + S^2 \quad (17)$$

It can be seen easily that $a < 0$ and $c > 0$ for this case. Then we can conclude that the discriminant of equation (16) is $(b-B_i)^2 - 4ac > 0$ and have one solution

$$u_i = \frac{-(b-B_i)-\sqrt{(b-B_i)^2-4ac}}{2a}$$

which is positive.

For all cases above we have unique solution of (16) for $k = 1$.

Let $k = 2$. Then

$$B_i u_i = \left((2-2\theta-\theta_1)u_i^2 + (\theta^2\theta_1 + (A+2S)(\theta-1))u_i + 1 + (\theta_1-1)B + AS + S^2\right)^2. \quad (18)$$

Case $\theta = \theta_1 = 1$. From (18) we rewrite

$$B_i u_i = (-u_i^2 + u_i + 1 + AS + S^2)^2. \quad (19)$$

We will find critical points of a function $f(u_i) = (-u_i^2 + u_i + 1 + AS + S^2)^2$:

$$u_{i_1} = \frac{1}{2}, \quad u_{i_2} = \frac{1+\sqrt{1+4(1+AS+S^2)}}{2} > 0, \quad u_{i_3} = \frac{1-\sqrt{1+4(1+AS+S^2)}}{2} < 0.$$

Since that the left side of equation (19) is linear increasing function, we can conclude that equation (18) has two positive solutions.

Case $\theta = 1$ **and** $\theta_1 > 1$. As the above case, it is also used critical points to analyze the number of solutions of (18) for this case:

$$B_i u_i = \left(-\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1-1)B\right)^2$$

and

$$f'(u_i) = \left(\left(-\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1-1)B\right)^2\right)' = 0.$$

We have

$$u_{i_1} = \frac{1}{2}, \quad u_{i_2} = \frac{\theta_1+\sqrt{\theta_1^2+4\theta_1(1+AS+S^2+(\theta_1-1)B)}}{2\theta_1} > 0,$$

$$u_{i_3} = \frac{\theta_1-\sqrt{\theta_1^2+4\theta_1(1+AS+S^2+(\theta_1-1)B)}}{2\theta_1} < 0.$$

and conclude that equation (18) has two positive solutions.

Case $\theta > 1$ **and** $\theta_1 \geq 1$. In this case using (17) we get the following critical points and two positive solutions of (18):

$$u_{i_1} = \frac{-b}{2a} > 0, \quad u_{i_2} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} > 0, \quad u_{i_3} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 0$$

We can say from the above cases, we have two solutions of (18) that are concluded using critical points for $k = 2$.

Now we will analyze equation (15) by dividing into two cases: k is odd and k is even separately.

Let k is odd. Then we will analyze function

$$f(u_i) = \left((2 - 2\theta - \theta_1)u_i^2 + (\theta^2\theta_1 + (A + 2S)(\theta - 1))u_i + 1 + (\theta_1 - 1)B + AS + S^2\right)^k. \tag{20}$$

Case $\theta = 1$ **and** $\theta_1 \geq 1$. In this case the following function from (20)

$$f(u_i) = \left(-\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1 - 1)B\right)^k$$

is increasing for $u_i < \frac{1}{2}$ and we conclude that (15) has unique positive solution.

Case $\theta > 1$ **and** $\theta_1 \geq 1$. In this case the function (20) is increasing for

$$u_i < -\frac{\theta^2\theta_1 + (A + 2S)(\theta - 1)}{2(2 - 2\theta - \theta_1)}$$

and equation (15) has unique positive solution.

Let k is even.

Case $\theta = 1$ **and** $\theta_1 \geq 1$. In this case the derivative of the equation (20) is

$$f'(u_i) = k(-2\theta_1 u_i + \theta_1)\left(-\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1 - 1)B\right)^{k-1}.$$

The function (20) is increasing when

$$\begin{cases} -2\theta_1 u_i + \theta_1 > 0 \\ -\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1 - 1)B > 0, \end{cases}$$

and

$$\begin{cases} -2\theta_1 u_i + \theta_1 < 0 \\ -\theta_1 u_i^2 + \theta_1 u_i + 1 + AS + S^2 + (\theta_1 - 1)B < 0 \end{cases}$$

systems are satisfied.

Solving the systems above, we have two positive solutions of (15) for $u_{i_1} \in (0, \frac{1}{2})$ and $u_{i_2} \in \left(\frac{\theta_1 + \sqrt{\theta_1^2 + 4\theta_1(1 + AS + S^2 + (\theta_1 - 1)B)}}{2\theta_1}; +\infty\right)$.

Case $\theta > 1$ **and** $\theta_1 \geq 1$. In this case, using the same method of the previous case, we can see the function (20) is increasing for $u_{i_1} \in (0, -\frac{b}{2a})$ and $u_{i_2} \in \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}; +\infty\right)$, since $a < 0, b > 0, c > 0$ where a, b and c are in (17).

Summarising, we obtain the following

Theorem 2. *If $\theta \geq 1$ and $\theta_1 \geq 1$, then for any $\beta > 0$ and any fixed probability measure ν on Φ the model* (1) *has*

(1) a unique translation-invariant splitting Gibbs measure if k is odd;

(2) two translation-invariant splitting Gibbs measures if k is even.

Remark 2. *In [13] we gave a description of the class of measures ν on Φ such that respect to each element of this class the infinite system of equations has unique solution $\{a^i, i = 1, 2, ...\}$, where $a \in (0, 1)$.*

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