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GIBBS MEASURES OF MODELS WITH UNCOUNTABLE SET OF SPIN VALUES ON LATTICE SYSTEMS

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Abstract

In this paper, we shall discuss the construction of Gibbs measures for models with uncountable set of spin values on Cayley trees. From [22] it is known that "translation-invariant Gibbs measures" of the model with an uncountable set of spin values can be described by positive fixed points of a nonlinear integral operator of Hammerstein type. The problem of constructing a kernel with non-uniqueness of the integral operator is sufficient in Gibbs measure theory. In this paper, we construct a degenerate kernel in which the number of solutions does not exceed 3, and in turn, it only gives us a chance to check the existence of phase transitions.

Keywords: *Cayley tree, cylinder sets, Kolmogorov's extension theorem, spin values, non-probability measures.*

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Introduction

Foundations of measure theory provide little support for compositional reasoning. Standard formalizations of iterative processes prefer to construct a single monolithic sample space from which all random choices are made at once. The central result in this regard is the Kolmogorov's extension theorem, which identifies conditions under which a family of measures on finite subproducts of an infinite product space extend to a measure on the whole space. This theorem is typically used to construct a large sample space for an infinite iterative process when the behavior of each individual step of the process is known (see [2, 19]).

Gibbs measures for models with an uncountable set of spin values on lattice systems requires some additional considerations compared to models with a countable set of spin values. The construction of Gibbs measures for models with an uncountable set of spin values on lattice systems is an active area of research in statistical mechanics and mathematical physics. Different techniques and methods have been developed to tackle specific models and situations, and there is still ongoing work to fully understand the properties and behavior of these measures (e.g. [22]).

In lattice systems with an uncountable set of spin values, such as real-valued spins, the traditional approach of defining the Gibbs measure as a product measure on the lattice may not be applicable. This is because the product measure may not be well-defined due to the infinite product of probability measures.

One common approach to construct Gibbs measures for such models is to use the concept of conditional probabilities. The idea is to define the Gibbs measure as a collection of conditional probability measures, where each spin value is assigned a conditional probability given the values of neighboring spins. (e.g., [13, 14]).

The paper deals with the problem of constructing kernels of Hammerstein-type equations whose positive solutions are not unique. This problem arises from the theory of Gibbs measures, and each positive solution of the equation corresponds to one translation-invariant Gibbs measure. Also, the problem of finding kernels for which the number of positive solutions to the equation is greater than one is equivalent to the problem of finding models which has phase transition. The problem of constructing kernels of the equation for which there are at least two positive solutions is also studied in [16, 10, 11, 12, 4]. In these articles, the number of solutions corresponding to the constructed kernels does not exceed 3, and in turn, it only gives us a chance to check the existence of phase transitions. The constructed kernels in this paper are more general than kernels in the above-mentioned papers and except for checking phase transitions, it allows us to classify the set of Gibbs measures.

1 Cylindric sets on Cayley trees

The Cayley tree $\Im^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e. graph without cycles, each vertex of which has exactly $k+1$ edges. Here V is the set of vertices of $\Im^k$ va L is the set of its edges.

Consider models where the spin takes values in the set Φ (finite or denumerable), and is assigned to the vertices of the tree. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to \Phi$. Let $\Omega_A = \Phi^A$ be the set of all configurations on A. A configuration σ on V is defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is $\Omega := \Phi^V$. We consider all elements of V are numerated (in any order) by the numbers: $0, 1, 2, 3, ...$ Namely, we can write $V = \{x_0, x_1, x_2, ...\}$ (detail in [3, 20, 21]).

Let $\mathcal{X}_A$ be the indicator function. Ω can be considered as a metric space with respect to the metric $\rho : \Omega \times \Omega \to \mathbb{R}^+$ given by

$$\rho\left(\{\sigma(x_n)\}_{x_n \in V}, \{\sigma'(x_n)\}_{x_n \in V}\right) = \sum_{n \geq 0} 2^{-n} \mathcal{X}_{\sigma(x_n) \neq \sigma'(x_n)}$$

(or any equivalent metric the reader might prefer, this metric taken from [13]), and let $\mathcal{B}$ be the σ-field of Borel subsets of Ω.

For each $m \geq 0$ let $\pi_m : \Omega \to \Phi^{m+1}$ be given by $\pi_m(\sigma_0, \sigma_1, \sigma_2, ...) = (\sigma_0, \ldots, \sigma_m)$ and let $\mathcal{C}_m = \pi_m^{-1}\left(\mathcal{P}\left(\Phi^{m+1}\right)\right)$, where $\sigma_i := \sigma(x_i)$ and $\mathcal{P}\left(\Phi^{m+1}\right)$ is the family of all subsets of Φ^{m+1} (Cartesian product of Φ). Then $\mathcal{C}_m$ is a field and each of the sets in $\mathcal{C}_m$ is open and closed set in the metric space (Ω, ρ); also $\mathcal{C}_m \subset \mathcal{C}_{m+1}$. Let $\mathcal{C} = \bigcup_{m \geq 0} \mathcal{C}_m$; then $\mathcal{C}$ is a field (the field of **cylinder sets**) and each of the sets in $\mathcal{C}$ is both open and closed. Denote $\mathcal{S}(\mathcal{C})$ - the smallest sigma field containing $\mathcal{C}$. Every element of

$\mathcal{S}(\mathcal{C})$ is called "**measurable cylinder**". Put

$$\sigma^{(m)}(q) = \left\{\sigma \in \Omega : \ \sigma\big|_{\{x_m\}} = q \in \Phi\right\}.$$

Definition 1. *A measurable space $(X, \mathcal{E})$ is said to be countably generated if $\mathcal{E} = \sigma(\mathcal{I})$ for some countable subset $\mathcal{I}$ of $\mathcal{E}$.*

Proposition 1. *[15] $\mathcal{B} = \mathcal{S}(\mathcal{C}) = \mathcal{S}\left(\left\{\sigma^{(m)}(q) : m \geq 0, \ q \in \Phi\right\}\right)$ and in particular if $|\Phi| < \infty$ then $(\Omega, \mathcal{B})$ is countably generated.*

Proof. Let $\mathcal{O}$ be the set of all open subsets of Ω. Then $\mathcal{C}$ (if $|\Phi| < \infty$ then $\mathcal{C}$ is a countable set) a base for the topology on the metric space (Ω, ρ). Also, since $\mathcal{C}$ is the field, each $O \in \mathcal{O}$ can be written as a union of elements from $\mathcal{C}$. Hence $\mathcal{O} \subset \mathcal{S}(\mathcal{C})$ and thus $\mathcal{B} = \mathcal{S}(\mathcal{O}) \subset \mathcal{S}(\mathcal{S}(\mathcal{C})) = \mathcal{S}(\mathcal{C})$, i.e., $\mathcal{B} = \mathcal{S}(\mathcal{C})$. Moreover, each element of $\mathcal{C}$ can be written as a finite intersection of elements from the set $\left\{\sigma^{(m)}(q) : m \geq 0, \ q \in \Phi\right\}$ and it therefore follows that

$$\mathcal{C} \subset \mathcal{S}\left(\left\{\sigma^{(m)}(q) : m \geq 0, \ q \in \Phi\right\}\right).$$

This implies that $\mathcal{B} = \mathcal{S}\left(\left\{\sigma^{(m)}(q) : m \geq 0, \ q \in \Phi\right\}\right)$. □

For a fixed $x^0 \in V$ we put

$$W_n = \left\{x \in V \mid d\left(x, x^0\right) = n\right\}, \quad V_n = \bigcup_{m=0}^{n} W_m,$$

where $d(x, y)$ is the distance between the vertices x and y on the Cayley tree, i.e. the number of edges of the shortest walk (i.e., path) connecting vertices x and y.

For any fixed configuration $\sigma_A \in \Omega_A, \ A \subset V$ we denote:

$$\bar{\sigma}_A := \left\{\sigma \in \Omega : \ \sigma\big|_A = \sigma_A\right\}.$$

Corollary 1. *[15] $\mathcal{B} = \mathcal{S}\left(\{\bar{\sigma}_{V_n} : n \in \mathbb{N}_0\}\right)$.*

Proof. By Proposition 1 we have

$$\bar{\sigma}_{V_n} = \bigcap_{s_i \in V_n} \sigma^{(s_i)}(\bar{\sigma}_{V_n}(s_i)) \in \mathcal{S}(\{\sigma^{(m)}(q) : m \geq 0, \ q \in \Phi\}) = \mathcal{B}.$$

Then for all $n \in \mathbb{N}$ we obtain that $\bar{\sigma}_{V_n} \in \mathcal{B}$, i.e. $\mathcal{S}\left(\{\bar{\sigma}_{V_n} : n \in \mathbb{N}\}\right) \subseteq \mathcal{B}$.

On the other hand, we show that $\mathcal{B} \subseteq \mathcal{S}\left(\{\bar{\sigma}_{V_n} : n \in \mathbb{N}\}\right)$. Let $m_0 \geq 0$ and we can find $n_0 \in \mathbb{N}$ such that $x_{m_0} \in V_{n_0}$. If bases of cylinder sets $\bar{\omega}_{V_{n_0}}, \bar{\nu}_{V_{n_0}}$ coincide with each other only at $\{x_{m_0}\}$ and its value be $q_0 \in \Phi$ then we obtain that

$$\sigma^{(m_0)}(q_0) = \bar{\omega}_{V_{n_0}} \cap \ \bar{\nu}_{V_{n_0}} \in \mathcal{S}\left(\{\bar{\sigma}_{V_n} : n \in \mathbb{N}\}\right).$$

From m_0 and q_0 are arbitrary numbers and Proposition 1 we can conclude that $\mathcal{B} \subseteq \mathcal{S}\left(\{\bar{\sigma}_{V_n} : n \in \mathbb{N}\}\right)$. □

Note that Corrollary 1 is very important in the theory of Gibbs measures (see [20, 21]) and a family of sets $\{V_n\}_{n=1}^{\infty}$ is also **cofinal** sets [14].

2 Lattice system has Kolmogorov property

In this section, we use notations of the previous sections.

Definition 2. *For each $\Lambda \in \mathcal{N}$ let μ_Λ be a measure. The family of measures $\{\mu_\Lambda\}_{\Lambda\in\mathcal{N}}$ is said to be* ***consistent (compatible)*** *if $\mu_\Lambda(F_\Lambda) = \mu_\Delta(F_\Delta)$ for all $F_\Lambda = F_\Delta \in \mathcal{B}_\Lambda$ whenever $\Lambda \subset \Delta$.*

There are several versions of Kolmogorov's extension theorem for probability measures. But some problems reduced to Kolmogorov's theorem for non-probability measures (e.g., [18]). Actually, we can not apply the theorem for any infinite measures. But in this section we give Kolmogorov's extension theorem for a certain class of infinite measures.

Note that $\Omega_n = \Omega_{V_n}$ and $\mathcal{B}_n$ is the σ-ring of all Borel sets of Ω_n. Also, μ_n is a measure on $(\Omega_n, \mathcal{B}_n)$, $n \in \mathbb{N}$.

If all A_n can be chosen in $\pi_m^{-1}(\mathcal{B}_m)$ for fixed m, then we have

Theorem 1. *[15] If one of measures $\{\mu_n\}_{n=1}^{\infty}$, say μ_{n_0}, is σ-finite, then $\{\mu_n\}_{n=1}^{\infty}$ can be extended uniquely to a σ-additive measure on $\mathcal{B}$.*

Let Ω be a non-empty set and $(\Omega, \subseteq)$ is a partially ordered set (poset). The poset Ω is said to be ***complete*** if each non-empty subset of Ω possesses a least upper bound. If $\{z_n\}_{n\geq 1}$ is an increasing sequence of elements from Ω then $\lim_n z_n := \sup(\{z_n : n \geq 1\})$. So $\Sigma = \left(V, \mathcal{N}, \Omega, \{\mathcal{F}^\Lambda\}_{\Lambda\in\mathcal{N}}\right)$ is a lattice system. Namely, $\mathcal{N}$ equipped with a directed, countably generated partial order $\subseteq$, and a decreasing family $\mathbb{F} = \{\mathcal{F}_\Lambda\}_{\Lambda\in\mathcal{N}}$ of sub-σ-algebras of $\mathcal{F}$.

Suppose for each $\Lambda \in \mathcal{N}$ we have a strict $\mathcal{F}_\Lambda$-measurable quasi-probability kernel $\zeta_\Lambda \in K(\mathcal{F}_\Lambda)$. Then the family $\mathcal{V} = \{\zeta_\Lambda\}_{\Lambda\in\mathcal{N}}$ will be called an $\mathbb{F}$-***specification*** if $\zeta_\Delta = \zeta_\Delta\zeta_\Lambda$ whenever $\Lambda, \Delta \in \mathcal{N}$ with $\Lambda \subseteq \Delta$. Let $\mathcal{V} = \{\zeta_\Lambda\}_{\Lambda\in\mathcal{N}}$ be an $\mathbb{F}$-specification; then a probability measure $\mu \in \mathrm{P}(\mathcal{F})$ is called a ***Gibbs state with specification $\mathcal{V}$*** if $\mu = \mu\zeta_\Lambda$ for each $\Lambda \in \mathcal{N}$. Note that this definition of Gibbs states originates from Dobrushin [5, 6, 7, 8], and Lanford and Ruelle [17, 23].

Definition 3. *Let $P_\Lambda : \Omega \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be $\mathcal{F}_\Lambda$-measurable mapping for all $\Lambda \in \mathcal{N}$, then the collection $P = \{P_\Lambda\}_{\Lambda\in\mathcal{N}}$ is called* ***a potential****. Also, the following expression*

$$H_{\Delta,P}(\sigma) \stackrel{def}{=} \sum_{\Delta\cap\Lambda\neq\varnothing, \Lambda\in\mathcal{N}} P_\Lambda(\sigma), \quad \forall\sigma \in \Omega. \tag{1}$$

is called ***Hamiltonian*** *H associated to the potential P.*

Put

$$r(P) \stackrel{\text{def}}{=} \inf\{R > 0 : P_\Lambda \equiv 0 \text{ for all } \Lambda \text{ with } \operatorname{diam}(\Lambda) > R\}.$$

If $r(P) < \infty$, P has finite range and $H_{\Delta;P}$ is well defined. If $r(P) = \infty$, P has infinite range and, for the Hamiltonian to be well defined, we will assume that P is absolutely summable in the sense that

$$\sum_{\Lambda\in\mathcal{N}, x\in\Lambda} \|P_\Lambda\|_\infty < \infty, \quad \forall x \in V,$$

(remember that $\|f\|_\infty \overset{\text{def}}{=} \sup_\omega |f(\omega)|\Big)$ which ensures that the interaction of a spin with the rest of the system is always bounded, and therefore that $\|H_{\Delta;P}\|_\infty < \infty$.

Let $\mathcal{N}_1 = \{V_n : n \in \mathbb{N}\}$ then we define the following Hamiltonian in the box $V_n, n \in \mathbb{N}$:

$$H_n(\sigma) = \sum_n P_{V_n}(\sigma), \quad \forall \sigma \in \Omega. \tag{2}$$

Let us define a specification $\zeta^H = \left\{\zeta^H_{V_n}\right\}_{n\in\mathbb{N}}$ (in short $\zeta^H_{V_n} := \zeta^H_n$) such that $\zeta^H_{V_n}(\cdot \mid \omega)$ gives to each configuration $\tau_{V_n}\omega^c_{V_n}$ a probability proportional to the Boltzmann weight prescribed by equilibrium statistical mechanics (see e.g. [13]):

$$\zeta^P_n(\omega, \sigma_n) \overset{\text{def}}{=} \frac{1}{\mathbf{Z}^\omega_n} e^{-H_n\left(\sigma_n\omega_{\bar{V}_n}\right)}, \tag{3}$$

where we have written explicitly the dependence on $\omega_{\bar{V}_n}$, and $\mathbf{Z}^\omega_n$ is a partition function, i.e.,

$$\mathbf{Z}^\omega_n \overset{\text{def}}{=} \sum_{\sigma_n\in\Omega_{V_n}} \exp\left(-H_n\left(\sigma_n\omega_{\bar{V}_n}\right)\right).$$

Theorem 2. *If* $\mathbb{F}_1 := \left\{\zeta^P_n\right\}_{n\in\mathbb{N}}$ *then* $\mathbb{F}_1$ *is a specification.*

Proof. For a fixed $m, n \in \mathbb{N}$ with $V_n \subset V_m \Subset V$, we show that $\zeta^P_m\zeta^P_n = \zeta^P_m$. Let $A \subset \mathcal{P}(\Omega_{V_n})$. At first, we consider the case $A = \{\sigma_{V_n}\}$, i.e.,

$$\zeta_m\zeta_n(\omega, \sigma_{V_n}) = \sum_{\tau_{V_m}} \zeta_m\left(\omega, \tau_{V_m}\right)\zeta_n\left(\tau_{V_m}\omega_{V^c_m}, \sigma_{V_n}\right) = \sum_{\tau_{V_m}} \zeta_m\left(\omega, \tau_{V_m}\right)\zeta_n\left(\tau_{V_m\setminus V_n}\omega_{V^c_m}, \sigma_{V_n}\right).$$

From the last equation, for any $A \subset \mathcal{P}(\Omega_{V_n})$ we obtain

$$\zeta_m\zeta_n(A \mid \omega) = \sum_{\tau_{V_m}}\sum_{\eta_\Delta} \mathbf{1}_A\left(\eta_{V_n}\tau_{V_m\setminus V_n}\omega_{V^c_m}\right)\zeta_m\left(\omega, \tau_{V_m}\right)\zeta_n\left(\tau_{V_m\setminus V_n}\omega_{V^c_m}, \eta_{V_n}\right).$$

For any configuration τ_{V_m} which defined on V_m we rewrite this configuration as a combination of τ'_{V_n} on V_n and $\tau_{V_m\setminus V_n}$ on $V_m\setminus V_n$, i.e. $\tau_{V_m} = \tau'_{V_n} \vee \tau''_{V_m\setminus V_n}$. By (3), RHS of the last equation can be rewritten as:

$$\sum_{\tau''_{V_m\setminus V_n}}\sum_{\eta_{V_n}} \mathbf{1}_A\left(\eta_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right)\frac{e^{-H_{V_n}\left(\eta_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right)}}{\mathbf{Z}_m\left(\omega_{V^c_m}\right)\mathbf{Z}_n\left(\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right)}\sum_{\tau'_{V_n}} e^{-H_{V_m}\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right)}.$$

It's easy to check that the following expression

$$H_m\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right) - H_n\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right)$$

does not depend on τ'_{V_n}. That's why we have

$$H_m\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right) - H_n\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V^c_m}\right) = H_m\left(\eta_{V_n}\tau_{V_m\setminus V''_n}\omega_{V^c_m}\right) - H_n\left(\eta_{V_n}\tau_{V_m\setminus V''_n}\omega_{V^c_m}\right),$$

which gives

$$\sum_{\tau'_{V_n}} e^{-H_m\left(\tau'_{V_n}\tau''_{V_m\setminus V_n}\omega_{V_m^c}\right)} = \mathbf{Z}_n\left(\tau''_{V_m\setminus V_n}\omega_{V_m^c}\right) e^{H_m\left(\eta_{V_n}\tau''_{V_m\setminus V_n}{}^{\omega c}\right)} e^{-H_n\left(\eta_{V_n}\tau''_{V_m\setminus V_n}\omega_{V_m^c}\right)}.$$

Inserting this in the above expression, and renaming $\eta_{V_n}\tau''_{V_m\setminus V_n} \equiv \eta'_{V_m}$, we get

$$\zeta_m\zeta_n(A \mid \omega) = \sum_{\eta'_{V_m}} \mathbf{1}_A\left(\eta'_{V_m}\omega_{V_m^c}\right) \frac{e^{-H_m\left(\eta'_{V_m}\omega_{V_m c}\right)}}{\mathbf{Z}_m\left(\omega_{V_m^c}\right)} = \zeta_m(A \mid \omega).$$

□

3 Gibbs measures for models with uncountable set of spin values

For each $m \geq 0$ let $\pi_m : \Omega \to [0,1]^{m+1}$ be given by $\pi_m(\sigma_0, \sigma_1, \sigma_2, ...) = (\sigma_0, \ldots, \sigma_m)$ and let $\mathcal{C}_m = \pi_m^{-1}\left(\mathcal{P}\left([0,1]^{m+1}\right)\right)$, where $\sigma_i := \sigma(x_i)$ and $\mathcal{P}\left([0,1]^{m+1}\right)$ is the family of all subsets of $[0,1]^{m+1}$ (Cartesian product of $[0,1]$). Then $\mathcal{C}_m$ is a field and each of the sets in $\mathcal{C}_m$ is open and closed set in the metric space (Ω, ρ); also $\mathcal{C}_m \subset \mathcal{C}_{m+1}$. Let $\mathcal{C} = \bigcup_{m\geq 0}\mathcal{C}_m$; then $\mathcal{C}$ is a field (the field of **cylinder sets**) and each of the sets in $\mathcal{C}$ is both open and closed. Denote $\mathcal{S}(\mathcal{C})$ - the smallest sigma field containing $\mathcal{C}$. Every element of $\mathcal{S}(\mathcal{C})$ is called "**measurable cylinder**".

Let us consider a formal Hamiltonian:

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \quad \sigma \in \Omega_V \tag{4}$$

where $J \in R \setminus \{0\}$ and $\xi : (u,v) \in [0,1]^2 \to \xi_{uv} \in R$ is a given bounded, measurable function. As usual, $\langle x, y\rangle$ stands for the nearest neighbor vertices.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma_n(x),x}\right), \tag{5}$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x\in W_n} h_{\widetilde{\sigma}_n(x),x}\right) \lambda_{V_n}(d\widetilde{\sigma}_n). \tag{6}$$

Let $\Lambda \in \mathcal{N}$ and $\Delta \subset \Lambda$. If μ_Λ is a measure on $\mathcal{B}_\Lambda$, the projection of μ_Λ on $\mathcal{B}_\Delta$ is measure $\pi_\Delta(\mu_\Lambda)$ on $\mathcal{B}_\Delta$ defined by

$$[\pi_\Delta(\mu_\Lambda)](B) = \mu_\Lambda\{\sigma \in \Omega_\Lambda : \ \sigma|_\Delta \in B\}, \ B \in \mathcal{B}_\Delta.$$

Similarly, if μ is a measure on $\mathcal{B}$, the projection of μ on $\mathcal{B}_\Lambda$ is defined by

$$[\pi_\Lambda(\mu)](B) = \mu\{\sigma \in \Omega : \sigma_\Lambda \in B\} = \mu(\bar{\sigma}|_\Lambda = \sigma_\Lambda : \sigma_\Lambda \in B), \quad B \in \mathcal{B}_\Lambda.$$

The following theorem is known:

Theorem 3. *[1](**Kolmogorov Extension Theorem**) For each t in the arbitrary index set T, let Ω_t be a complete, separable metric space, with $\mathcal{F}_t$ the class of Borel sets (the σ-field generated by the open sets).*

Assume that for each finite nonempty subset v of T, we are given a probability measure P_v on $\mathcal{F}_v$. Assume the P_v are consistent, that is, $\pi_u(P_v) = P_u$ for each nonempty $u \subset v$.

Then there is a unique probability measure P on $\mathcal{F} = \prod_{t\in T} \mathcal{F}_t$ such that $\pi_v(P) = P_v$ for all v.

The probability distributions $\mu^{(n)}$ are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\pi_{V_{n-1}}\left(\mu^{(n)}\right) = \mu^{(n-1)} \tag{7}$$

Then by Kolmogorov extension theorem, there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$.

The measure μ is called *splitting Gibbs measure* corresponding to Hamiltonian (4) and function $x \mapsto h_x$, $x \neq x^0$.

Proposition 2. *[22] The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (5) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$f(t,x) = \prod_{y\in S(x)} \frac{\int_0^1 \exp(J\beta\xi_{tu}) f(u,y)du}{\int_0^1 \exp(J\beta\xi_{0u}) f(u,y)du}. \tag{8}$$

Here, and below $f(t,x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0,1]$ and $du = \lambda(du)$ is the Lebesgue measure.

Note, that the analysis of solutions to (8) is not easy. It's difficult to give a full description for the given potential function $\xi_{t,u}$.

Let ξ_{tu} is a continuous function. We put

$$C^+[0,1] = \{f \in C[0,1] : f(x) \geq 0\}, \;\; C_0^+[0,1] = C^+[0,1] \setminus \{\theta \equiv 0\}.$$

Define the operator $R_k : C_0^+[0,1] \to C_0^+[0,1]$ by

$$(R_k f)(t) = \left(\frac{\int_0^1 K(t,u)f(u)du}{\int_0^1 K(0,u)f(u)du}\right)^k, \;\; k \in \mathbb{N},$$

where $K(t,u) = \exp(J\beta\xi_{tu}), f(t) > 0, t, u \in [0,1]$.

We'll study the equation (8) in the class of translational-invariant functions $f(t,x)$, i.e $f(t,x)=f(t)\in C[0,1]$ for any $x\in V$ and it can be written as

$$(R_kf)(t)=f(t), \tag{9}$$

Note that equation (9) is not linear for any $k\geq 1$. For every $k\in\mathbb{N}$ we consider an integral operator H_k acting in the cone $C^+[0,1]$ i.e.,

$$(H_kf)(t)=\int_0^1 K(t,u)f^k(u)du, \quad k\in\mathbb{N}. \tag{10}$$

The operator H_k is called Hammerstein's integral operator of order k. Clearly, if $k\geq 2$ then H_k is a nonlinear operator.

Lemma 1. *[9] Let $k\geq 2$. The equation*

$$R_kf=f, \quad f\in C_0^+[0,1] \tag{11}$$

has a nontrivial positive solution iff the Hammerstein's operator has a positive eigenvalue, i.e. the Hammerstein's equation

$$H_kf=\lambda f, \quad f\in C^+[0,1] \tag{12}$$

has a nonzero positive solution for some $\lambda>0$.

It is easy to check that if the number $\lambda_0>0$ is an eigenvalue of the operator H_k, then an arbitrary positive number is an eigenvalue of the operator H_k (see Theorem 3.7 [9]), where $k\geq 2$. Consequently, we obtain

Let $f(t,x)$ does not depend on the vertices of the Cayley tree. Then the last equation (8) has a strongly positive solution if and only if the Hammerstein equation has a strongly positive solution in $\mathcal{M}_0=\{f\in C^+[0,1]: f(0)=1\}$, where $C^+[0,1]$ is the set of all positive continuous functions on $[0,1]$ (see [9]).

Let $f\left(\frac{1}{3}\right)=g\left(\frac{2}{3}\right)=c$ and denote that

$$\varphi_1(t)=\begin{cases} f(t) & \text{if } t\in\left[0,\frac{1}{3}\right] \\ c & \text{if } t\in\left[\frac{1}{3},\frac{2}{3}\right] \\ g(t) & \text{if } t\in\left[\frac{2}{3},1\right] \end{cases} \text{ and } \quad \varphi_2(t)=\begin{cases} g(1-t) & \text{if } t\in\left[0,\frac{1}{3}\right] \\ c & \text{if } t\in\left[\frac{1}{3},\frac{2}{3}\right] \\ f(1-t) & \text{if } t\in\left[\frac{2}{3},1\right] \end{cases}.$$

Also, for $f_1\left(\frac{1}{3}\right)=g_1\left(\frac{2}{3}\right)=c_1$, we define functions:

$$\psi_1(u)=\begin{cases} f_1(u) & \text{if } u\in\left[0,\frac{1}{3}\right] \\ c_1 & \text{if } u\in\left[\frac{1}{3},\frac{2}{3}\right] \\ g_1(u) & \text{if } u\in\left[\frac{2}{3},1\right] \end{cases} \text{ and } \quad \psi_2(u)=\begin{cases} g_1(1-u), & \text{if } u\in\left[0,\frac{1}{3}\right] \\ c_1, & \text{if } u\in\left[\frac{1}{3},\frac{2}{3}\right] \\ f_1(1-u), & \text{if } u\in\left[\frac{2}{3},1\right] \end{cases}.$$

By using these functions we define a degenerate kernel:

$$\tilde{K}(t,u)=\varphi_1(t)\psi_1(u)+\varphi_2(t)\psi_2(u). \tag{13}$$

Then the equation (10) can be written as

$$f(t) = \int_0^1 (\varphi_1(t)\psi_1(u) + \varphi_2(t)\psi_2(u)) f^k(u)du. \tag{14}$$

Namely,

$$f(t) = \varphi_1(t) \int_0^1 \psi_1(u) f^k(u)d(u) + \varphi_2(t) \int_0^1 \psi_2(u) f^k(u)d(u) = f(t).$$

Put

$$C_1 = \int_0^1 \psi_1(u) f^k(u)d(u), \qquad C_2 = \int_0^1 \psi_2(u) f^k(u)du.$$

Consequently, by taking into account $f(t) = C_1\varphi_1(t) + C_2\varphi_2(t)$ we obtain

$$C_i = \int_0^1 \psi_i(u)(C_1\varphi_1(u) + C_2\varphi_2(u))^k du, \quad i \in \{1, 2\}.$$

The last equality is equivalent to

$$\begin{cases} C_1 = \binom{k}{0} C_1^k \alpha_1 + \binom{k}{1} C_1^{k-1} C_2 \alpha_2 \quad + ... \quad + \binom{k}{k} C_2^k \alpha_{k+1} \\ C_2 = \binom{k}{0} C_2^k \alpha_1 + \binom{k}{1} C_2^{k-1} C_1 \alpha_2 \quad + ... \quad + \binom{k}{k} C_1^k \alpha_{k+1}. \end{cases} \tag{15}$$

Here and below,

$$\alpha_1 = \int_0^1 \psi_1(u)\varphi_1^k(u)du, \quad \alpha_2 = \int_0^1 \psi_1(u)\varphi_1^{k-1}(u)\varphi_2(u)du, \ ... \ \alpha_{k+1} = \int_0^1 \psi_1(u)\varphi_2^{k+1}(u)du,$$

$$\beta_1 = \int_0^1 \psi_2(u)\varphi_1^k(u)du, \quad \beta_2 = \int_0^1 \psi_2(u)\varphi_1^{k-1}(u)\varphi_2(u)du, \ ... \ \beta_{k+1} = \int_0^1 \psi_2(u)\varphi_2^{k+1}(u)du. \tag{16}$$

Let $x = \frac{C_1}{C_2}$ then the system of equations (15) can be written as

$$x = \frac{\binom{k}{0}\alpha_1 x^k + \binom{k}{1}\alpha_2 x^{k-1} + \binom{k}{2}\alpha_3 x^{k-2} + \ ... \ + \binom{k}{k-1}\alpha_k x + \binom{k}{k}\alpha_{k+1}}{\binom{k}{0}\alpha_{k+1} x^k + \binom{k}{1}\alpha_k x^{k-1} + \binom{k}{2}\alpha_{k-1} x^{k-2} + \ ... \ + \binom{k}{k-1}\alpha_2 x + \binom{k}{k}\alpha_1}.$$

Namely,

$$Q_k(x) := \begin{pmatrix} k \\ 0 \end{pmatrix} \alpha_{k+1} x^{k+1} + \left(\begin{pmatrix} k \\ 1 \end{pmatrix} \alpha_k - \begin{pmatrix} k \\ 0 \end{pmatrix} \alpha_1 \right) x^k + ...$$

$$... + \left(\begin{pmatrix} k \\ k \end{pmatrix} \alpha_1 - \begin{pmatrix} k \\ k-1 \end{pmatrix} \alpha_k \right) x - \begin{pmatrix} k \\ k \end{pmatrix} \alpha_{k+1}. \tag{17}$$

Now, we can conclude the following result:

Proposition 3. *Finding positive solutions of the equation (10) with the kernel (13) is equivalent to finding positive roots of the polynomial* $Q_k(x)$.

Now, we study positive solutions to the equation (10) with the kernel (13) for the case $k = 2$. Let $k = 2$ then the system of equations (15) can be rewritten as

$$\begin{cases} C_1 = C_1^2 \alpha_1 + 2C_1C_2\alpha_2 + C_2^2\alpha_3 \\ C_2 = C_1^2 \beta_1 + 2C_1C_2\beta_2 + C_2^2\beta_3. \end{cases} \tag{18}$$

By (16) and construction of kernel one gets:

$$\alpha_1 = \int_0^1 \psi_1(u)\varphi_1^2(u)d(u) = \int_0^{\frac{1}{3}} f_1(u)f^2(u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) + \int_{\frac{2}{3}}^1 g_1(u)g^2(u)du =$$

$$= \int_0^{\frac{1}{3}} g_1(1-u)g^2(1-u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) +$$

$$+ \int_{\frac{2}{3}}^1 f_1(1-u)f^2(1-u)du = \int_0^1 \psi_2(u)\varphi_2^2(u)d(u) = \beta_3.$$

Similarly,

$$\alpha_2 = \int_0^1 \psi_1(u)\varphi_1(u)\varphi_2(u)d(u) = \int_0^{\frac{1}{3}} f_1(u)f(u)g(1-u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) +$$

$$+ \int_{\frac{2}{3}}^1 g_1(u)g(u)f(1-u)du = \int_0^{\frac{1}{3}} g_1(1-u)f(u)g(1-u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) +$$

$$+ \int_{\frac{2}{3}}^1 f_1(1-u)g(u)f(1-u)d(u) = \int_0^1 \psi_2(u)\varphi_1(u)\varphi_2(u)d(u) = \beta_2$$

and

$$\alpha_3 = \int_0^1 \psi_1(u)\varphi_2^2(u)d(u) = \int_0^{\frac{1}{3}} f_1(u)g^2(1-u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) +$$

$$+ \int_{\frac{2}{3}}^1 g_1(u)f^2(1-u)d(u) = \int_0^{\frac{1}{3}} g_1(1-u)f^2(u)d(u) + \int_{\frac{1}{3}}^{\frac{2}{3}} c_1c^2 d(u) +$$

$$+ \int_{\frac{2}{3}}^1 f_1(1-u)g^2(u)d(u) = \int_0^1 \psi_2(u)\varphi_1^2(u)d(u) = \beta_1.$$

Hence

$$\alpha_1 = \beta_3, \quad \alpha_2 = \beta_2, \quad \alpha_3 = \beta_1. \tag{19}$$

By taking into account $x = \frac{C_1}{C_2}$ and the equality (17) we have

$$Q_2(x) = \alpha_3 x^3 + (2\alpha_2 - \alpha_1)x^2 + (\alpha_1 - 2\alpha_2)x - \alpha_3 = 0.$$

Thus, we obtain

$$Q_2(x) = (x-1)(a_3x^2 + (a_3 + 2a_2 - a_1)x + a_3) = 0.$$

Put

$$D := (2a_2 - a_1 - a_3)(2a_2 - a_1 + 3a_3).$$

It is easy to check $a_1 + a_3 \geq 2a_2$. Therefore, the sign of D is the same as the sign of the expression $a_1 - 2a_2 - 3a_3$.

Proposition 4. *Let $k = 2$ and $\tilde{K}(t,u)$ be the kernel which defined in (13). Then the following statements hold:*

1. *If $a_1 < 2a_2 + 3a_3$ then there is a unique positive solution of (14);*
2. *If $a_1 = 2a_2 + 3a_3$ then there are exactly two positive solutions of (14);*
3. *If $a_1 > 2a_2 + 3a_3$ then there are exactly three positive solutions of (14).*

In the language of Gibbs measure theory we obtain:

Theorem 4. *Let $k = 2$ and $\tilde{K}(t,u)$ be the function of the Hamiltonian (4). Then the following assignments hold:*

1. *If $a_1 < 2a_2 + 3a_3$ then there is a unique translation-invariant Gibbs measure of the model (4);*
2. *If $a_1 = 2a_2 + 3a_3$ then there are exactly two translation-invariant Gibbs measures of the model (4);*
3. *If $a_1 > 2a_2 + 3a_3$ then there are exactly three translation-invariant Gibbs measures of the model (4).*

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