A new capacity in the class of sh_m functions defined by laplace operator

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Abstract

In this paper, we define a new capacity Δ_m on the class of sh_m functions, which is defined by Laplace operator. We prove that Δ_m-capacity satisfies Choquet's axioms of measurability. Moreover, we compare our capacity with Sadullaev-Abdullaev capacities. In particular, it implies that Δ_m-capacity of a set E is zero if and only if E is a m-polar set.

Keywords: *strongly m-subharmonic functions, Laplace operator, capacity, condenser capacity, external capacity, m-convex domain, m-polar set, m-capacity.*

Mathematics Subject Classification (2020): *31B05, 31B15.*

1 Introduction

Capacity is a set function arising in (pluri)potential theory as the analogue of the physical concept of the electrostatic capacity. In this paper, we study capacities in $\mathbb{C}^n$. Capacities actively studied by many authors Sadullaev (see [1-4]), Bedford and Taylor (see [6]), Abdullaev (see [1]), Rakhimov (see [4,8,9]) and other mathematicians. There are several capacities in the class of subharmonic and plurisubharmonic functions in $\mathbb{C}^n$: capacity of condenser, P-capacity, Δ-capacity (see [1-8]). Moreover, similar capacities is defined in the class of strongly $m-$subharmonic (sh_m) functions and actively studied by Sadullaev and Abdullaev (see [1]). In this work, we will define a new capacity on the class of sh_m functions defined by Laplace operator $\Delta u = dd^c u \wedge \beta^{n-1}$, where $\beta = dd^c|z|^2$. Since it is a linear operator this operator is more practical in use.

Our paper is organized as follows: in Section 2 we recall some notions and theorems, and moreover, we define a new capacity and prove some of its properties. In Section 3 we give definitions and some properties of $\mathbb{C}^n$-capacities defined by Sadullaev and Abdullaev. Finally, in Section 4 we compare our capacity with Sadullaev-Abdullaev capacities.

2 Δ_m-capacity and its properties

Firstly, let us recall some definitions and results from [1]. Let D be a domain in $\mathbb{C}^n$. For a real function $u \in C^2(D)$ the second order differential

$$dd^c u = \frac{i}{2} \sum_{j,k} u_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

(at a fixed point $z^0 \in D$) is a Hermitian quadratic form, where $u_{j\bar{k}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}$. After approaching a general unitary transformation of coordinates, it is reduced to the following diagonal form

$$dd^c u = \frac{i}{2}\big[\lambda_1 dz_1 \wedge d\overline{z}_1 + ... + \lambda_n dz_n \wedge d\overline{z}_n\big]$$

where $\lambda_1(u), ..., \lambda_n(u)$ are the eigenvalues of the Hermitian matrix $(u_{j,\overline{k}})$, which are real, i.e. $(\lambda_1(u), ..., \lambda_n(u)) \in \mathbb{R}^n$. It's not hard to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H_m(u)\beta^n, \ \forall k = 1, 2, ..., n,$$

where $H_k(u) = \sum_{1<j_1<...<j_k \leq n} \lambda_{j_1}...\lambda_{j_k}$ is Hessian of dimention k of vector $\lambda = \lambda(u) \in \mathbb{R}^n$ and $\beta = dd^c\|z\|^2$.

Definition 1. *(see [1]) Twice smooth function $u(z) \in C^2(D)$ is called strongly m-subharmonic at the point $z^0 \in D$, if:*

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \ \forall k = 1, 2, ..., n-m+1.$$

Now let us define strongly m-subharmonic functions in a larger class

Definition 2. *(see [1]) A function $u \in L^1_{loc}(D)$, is called strongly m-subharmonic (sh_m) in $D \subset \mathbb{C}^n$, if it is upper semicontinuous and for any twice smooth strongly m-subharmonic functions $v_1, ..., v_{m-1}$ the current $dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}$ is positive, where the current is defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}\right](\omega) = \int u dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \beta^{n-m} \wedge dd^c \omega$$

for any smooth function ω with compact support in D.

The class of strongly m-subharmonic (sh_m) functions on D denoted by $sh_m(D)$.

Let us remind that it is called strongly sh_m in the domain D, if $\rho(z)$ is *strongly sh_m* at every point $z^0 \in D$. Domain $D \subset \mathbb{C}^n$ is called *strongly m-convex*, if there exists a strongly sh_m function $\rho(z)$ in a neighbourhood G of $\overline{D}$, such that $D = \{\rho < 0\}$. Let us recall the notion of m-polarity, which plays the role of null measure sets for m-capacities.

Definition 3. *The set $E \subset D \subset \mathbb{C}^n$ is called m-polar in D, if there is a function $u(z) \in sh_m(D)$, $u(z) \not\equiv -\infty$ such that $u|_E = -\infty$.*

Δ_m-capacity. For simplicity, we work on a strongly $m-$convex domain $D \subset \mathbb{C}^n$. For a compact $K \subset\subset D$ define a class of functions

$$\mathcal{U}_m(K, D) = \left\{u(z) \in sh_m(D) \cap C(D): \ u|_K \leq -1, \, u|_D \leq 0, \ \varliminf_{z \to \partial D} u(z) \geq 0\right\}$$

and set the following quantity

$$\Delta_m(K,D) := \inf\left\{ \int\limits_D \Delta u : \quad u \in \mathcal{U}_m(K,D) \right\}.$$

The quantity $\Delta_m(K,D)$ is called Δ_m-*capacity* of K with respect to D.

From now on, we write $\Delta_m(K)$ instead of $\Delta_m(K,D)$ when the role of D is not important. The capacity $\Delta_m(K)$ has the following properties:

1°. *$\Delta_m(K)$ is monotone, i.e. for $K_1 \subset K_2$ we have $\Delta_m(K_1) \le \Delta_m(K_2)$.*

Proof. Thanks to $K_1 \subset K_2$, any function u, with $u|_{K_2} \le -1$, is clearly less -1 on K_1. Hence, we have $\mathcal{U}_m(K_1,D) \supset \mathcal{U}_m(K_2,D)$. Consequently, by definition we can easily see that $\Delta_m(K_1) \le \Delta_m(K_2)$.

2°. *$\Delta_m(K)$ is sub-additive, i.e. $\Delta_m(K_1 \cup K_2) \le \Delta_m(K_1) + \Delta_m(K_2)$.*

Proof. Clearly, if $u_1 \in \mathcal{U}_m(K_1,D)$ and $u_2 \in \mathcal{U}_m(K_2,D)$ then $u_1 + u_2 \in \mathcal{U}_m(K,D)$, where $K = K_1 \cup K_2$. Therefore

$$\begin{aligned}
\Delta_m(K) &= \inf\left\{ \int\limits_D \Delta u : \ u \in \mathcal{U}_m(K,D) \right\} \le \\
&\le \inf\left\{ \int\limits_D \Delta(u_1+u_2) : \ u_1 \in \mathcal{U}_m(K_1,D), u_2 \in \mathcal{U}_m(K_2,D) \right\} = \\
&= \inf\left\{ \int\limits_D \Delta u_1 + \int\limits_D \Delta u_2 : \ u_1(z) \in \mathcal{U}_m(K_1,D), u_2(z) \in \mathcal{U}_m(K_2,D) \right\} = \\
&= \inf\left\{ \int\limits_D \Delta u_1 : \ u_1 \in \mathcal{U}_m(K_1,D) \right\} + \\
&+ \inf\left\{ \int\limits_D \Delta u_2 : \ u_2 \in \mathcal{U}_m(K_2,D) \right\} = \Delta_m(K_1) + \Delta_m(K_2).
\end{aligned}$$

The proof of the property is complete.

3°. *$\Delta_m(K)$ is monotonic by m, i.e., $\Delta_1(K) \le \Delta_2(K) \le \ldots \le \Delta_n(K)$.*

Proof. Since $sh(D) = sh_1(D) \supset sh_2(D) \supset \ldots \supset sh_n(D) = psh(D)$ we have $\mathcal{U}_1(K,D) \supset \mathcal{U}_2(K,D) \supset \ldots \supset \mathcal{U}_n(K,D)$. Hence, we deduce that

$$\Delta_1(K,D) \le \Delta_2(K,D) \le \ldots \le \Delta_n(K,D).$$

Let us now define the external capacity in a standard way.

Definition 4. *Let E be a subset of D. The* external capacity *of E is*

$$\Delta_m^*(E) = \inf\left\{\Delta_m(U) : U \supset E - \textit{open set}\right\},$$

where capacity of open set $U \subset D$ is defined by

$$\Delta_m(U) = \sup\left\{\Delta_m(K) : K \subset\subset U\right\}.$$

Now we collect the following properties of the external capacity.

4°. *For any compact $K \subset\subset D$, the external capacity of K is equal to Δ_m-capacity of K, i.e., $\Delta_m^*(K) = \Delta_m(K)$.*

Proof. For any $\varepsilon > 0$ there exists an open set $U \supset K$, such that

$$\Delta_m^*(K) \geq \Delta_m(U) - \varepsilon \geq \Delta_m(K) - \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain

$$\Delta_m^*(K) \geq \Delta_m(K). \tag{1}$$

On the other hand, by definition of external capacity, we have

$$\Delta_m^*(K) \leq \Delta_m(U)$$

for all $U \supset K$. For any $\varepsilon > 0$ there exists $u \in \mathcal{U}_m(K, D)$ such that $\int_D \Delta u \leq \Delta_m(K) + \varepsilon$. Take an open set $K \subset U \Subset \{(1+\varepsilon)u < -1\}$. Since $(1+\varepsilon)u \in \mathcal{U}_m(\overline{U}, D)$ we have

$$\Delta_m(U) \leq \Delta_m(\overline{U}) \leq (1+\varepsilon)\int_D \Delta u \leq (1+\varepsilon)(\Delta_m(K) + \varepsilon).$$

So, we have

$$\Delta_m^*(K) \leq \Delta_m(U) \leq (1+\varepsilon)(\Delta_m(K) + \varepsilon).$$

By letting $\varepsilon \to 0$ we have

$$\Delta_m^*(K) \leq \Delta_m(K). \tag{2}$$

Thanks to (1) and (2) we have $\Delta_m^*(K) = \Delta_m(K)$ for any compact set $K \subset\subset D$. The proof is complete.

5°. *External capacity $\Delta_m^*(E)$ is monotonic, e.i.,*

$$E_1 \subset E_2 \quad \Rightarrow \quad \Delta_m^*(E_1) \leq \Delta_m^*(E_2).$$

Proof. Since $E_1 \subset E_2$ we have that

$$\Delta_m^*(E_2) = \inf\left\{\Delta_m(U) \ : \ U \supset E_2\right\} \geq \inf\left\{\Delta_m(U) \ : \ U \supset E_1\right\} = \Delta_m^*(E_1).$$

6°. $\Delta_m^*(E)$ *is countable-subadditive, e.i.* $\Delta_m^*\left(\bigcup_j E_j\right) \leq \sum_j \Delta_m^*(E_j)$.

Proof. From the definition, for any $\varepsilon > 0$ there are open subsets $U_j \supset E_j$ such that $\Delta_m(U_j) - \Delta_m^*(E_j) < \frac{\varepsilon}{2^j}$, $j = 1, 2, ...$ Then

$$\Delta_m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \Delta_m\left(\bigcup_{j=1}^{\infty} U_j\right) \leq \sum_{j=1}^{\infty} \Delta_m(U_j) \leq \Delta_m^*(E_j) + \varepsilon.$$

By letting $\varepsilon \to 0$, we get

$$\Delta_m^*\left(\bigcup_j E_j\right) \leq \sum_j \Delta_m^*(E_j).$$

7°. For any increasing sequence of sets $E_j \subset E_{j+1}$ the following equality holds

$$\Delta_m^*\left(\bigcup_j E_j\right) = \lim_{j\to\infty} \Delta_m^*(E_j).$$

Proof. Since E_j is an increasing sequence of sets, from the previous property, $\Delta_m^*(E_j)$ is also an increasing sequence and it has a limit as $n \to \infty$. Again, by the monotonicity of Δ_m^*, we have

$$\Delta_m^*\left(\bigcup_j E_j\right) \geq \Delta_m^*(E_k)$$

for all $k = 1, 2, \dots$. Hence

$$\Delta_m^*\left(\bigcup_j E_j\right) \geq \lim_{k\to\infty} \Delta_m^*(E_k).$$

Now let $E_j \subset E_{j+1}$ be arbitrary sets $j = 1, 2....$ Let us fix arbitrary number $\varepsilon > 0$. Then for every $j \in \mathbb{N}$ there exists open set U_j, with $U_j \subset U_{j+1}$ and such that $\Delta(U_j) - \Delta^*(E_j) < \sum_{l=1}^{j} \frac{\varepsilon}{2^l}$. Hence, for any k

$$\Delta_m\left(\bigcup_{j=1}^{k} U_j\right) \leq \Delta_m^*\left(\bigcup_{j=1}^{k} E_j\right) + \sum_{j=1}^{k} \frac{\varepsilon}{2^j}.$$

From here, by letting k go to infinity, we obtain

$$\Delta_m^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \Delta_m\left(\bigcup_j U_j\right) \leq \lim_{j\to\infty} \Delta^*(E_j) + \varepsilon$$

regardless of whether capacity $\Delta_m^*(E)$ finite or not. Since ε is arbitrary, the proof is complete.

The properties above guarantee that Δ_m^*-capacity is a Choquet capacity (see [7], page 64).

3 Other capacities in the class of sh_m functions

As above we fix $1 \le m \le n$ and a strongly m-convex domain $D \subset \mathbb{C}^n$. Let E be a subset of the domain D.

Definition 5 (see [1]). *Consider the class of functions*

$$\mathcal{U}(E,D) = \left\{ u(z) \in sh_m(D) : u|_D \le 0, u|_E \le -1 \right\}$$

and put

$$\omega(z,E,D) = \sup \left\{ u(z) : u \in \mathcal{U}(E,D) \right\}.$$

Then the regularization $\omega^*(z,E,D) = \overline{\lim_{w\to z}}\, \omega(w,E,D) = \lim_{\varepsilon\to 0} \sup_{w\in B(z,\varepsilon)} \omega(w,E,D)$ *is called the* $m-$*subharmonic measure* ($\mathcal{P}_m$*-measure) of* E *with respect to* D*, where* $B(a,\rho)$ *is a ball centred at* a *and radius* $\rho > 0$.

Let us now define a capacity defined by Sadullaev and Abdullaev.

Definition 6 (see [1]). *Let* $E \subset D$ *and* $\omega^*(z,E,D)$ *be its* $\mathcal{P}_m$*-measure. Then the integral*

$$\mathcal{P}_m(E,D) = -\int_D \omega^*(z,E,D)dV$$

is called the $\mathcal{P}_m$*-capacity of the set* E *with respect to* D.

The capacity $\mathcal{P}_m(E,D)$ is well studied. In particular, it is zero if and only if E is m-polar set. It is monotonic, countably subadditive and satisfies Choquet's axioms of measurability (see [1]).

The more natural concept is condenser capacity, which is defined using the Hessian as total mass of the measure $\left(dd^c\omega^*(z,K,D)\right)^m \wedge \beta^{n-m}$.

Definition 7 (see [1]). *Let* K *compact in* $D \subset \mathbb{C}^n$. *The following quantity*

$$C_m(K,D) = \inf\left\{ \int_D (dd^c u)^m \wedge \beta^{n-m} : u \in sh_m(D) \cap C(D), u|_K \le -1, \underline{\lim_{z\to \delta D}}\, u(z) \ge 0 \right\}$$

is called the condenser capacity (m-capacity) of (K,D).

Note that, for $E \subset D$ the external capacity $C_m^*(E,D)$ defined in a standard way. The capacity $C_m^*(E,D)$ is well studied and has all the properties of capacities (see [2]). In particular it is zero if and only if E is a m-polar set.

4 Main result

In this section we will compare Δ_m^*-capacity with other two capacities whose defined in the previous section.

Theorem 1. *The following statements are true:*

(i) Let $E \subset B(0,r) \subset\subset B(0,R)$, $r<R$. *Then*

$$\Delta_m^*(E,B(0,r)) \leq \frac{1}{(1+a(r))(R^2-r^2)}\mathcal{P}_m(E,B(0,R)),$$

where $a(r) = \sup\limits_{B(0,r)} \omega^*(z,E,B(0,R)) > -1$.

(ii) Let $E \subset D$ *, then there exists a constant* $M(D)>0$ *(depending on measure of* D*) such that*

$$\sqrt[m]{C_m^*(E,D)} \leq M(D)\cdot \Delta_m^*(E,D).$$

Proof. First of all, we will prove the first inequality (i). We can assume that E is a regular compact. Let $\rho(z)=|z|^2-R^2$ and $\omega^* := \omega^*(z,E,B(0,R)) = \omega(z,E,B(0,R)$. For the following inequality, we use the similar steps as in [8]

$$\int\limits_{-R^2}^{0} dt \int\limits_{\rho(z)\leq t} (dd^c\rho)^{n-1}\wedge dd^c\omega^* = \int\limits_{|z|=R} \omega^* d^c|z|^2 \wedge (dd^c|z|^2)^{n-1} =$$

$$= -\int\limits_{|z|\leq R} \omega^*(dd^c|z|^2)^n = \mathcal{P}_m(E,B(0,R)).$$

On the other hand, we can estimate the LHS of this inequality from below:

$$\int\limits_{-R^2}^{0} dt \int\limits_{\rho(z)\leq t} dd^c\omega^* \wedge (dd^c\rho)^{n-1} \geq \int\limits_{r^2-R^2}^{0} dt \int\limits_{\rho(z)\leq r^2} dd^c\omega^* \wedge (dd^c\rho)^{n-1} =$$

$$= \left(R^2-r^2\right) \int\limits_{\rho(z)\leq r^2} dd^c\omega^* \wedge (dd^c\rho)^{n-1} =$$

$$= \left(R^2-r^2\right) \int\limits_{\rho(z)\leq r} \Delta\omega^* \geq$$

$$\geq \left(R^2-r^2\right)\left(1+a(r)\right)\Delta_m^*(E,B(0,r)).$$

The last inequality follows due to regularity of E and $\frac{\omega^*(z,E,B(0,R))-a(r)}{1+a(r)} \in \mathcal{U}_m(E,B(0,r))$. So the assertion (i) is proved.

Now, we will prove the assertion (ii). We shall need the following claim.

Claim. *Let n, m be the natural numbers with $1 \leq m \leq n$. Suppose that, $\lambda_1, \lambda_2, ..., \lambda_n$ are real numbers and*

$$\sum_{1 \leq i_1 < ... < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \geq 0$$

for all $k = 1, 2, ..., m$. Then

$$C_k = \left(\frac{\lambda_1 + \lambda_2 + \ldots + \lambda_n}{n} \right)^m \geq \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m}.$$

Proof of the Claim. We take a polynomial with roots $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$P(x) = \prod_{k=1}^{n} (x - \lambda_k) = x^n + \ldots + a_1 x^1 + a_0,$$

where $a_{n-j} = (-1)^j C_j$. Observe that since $P(x)$ has n real roots $P'(x)$ has $n-1$ real roots counting with multiplicities. Note that $C_j \geq 0$. Hence, $(n-m)$'th derivative

$$P^{(n-m)}(x) = b_m x^m + \ldots + b_1 x^1 + b_0$$

has m non-negative roots, since $b_j = (-1)^{m-j}(j+1)(j+2)...(j+n-m)C_{m-j}$ for $0 \leq j \leq m-1$ and $b_m = (m-1)(m+1) \cdot \ldots \cdot n$. Denote the m non-negative roots of $P^{(n-m)}(x)$ by $x_1, \ldots, x_m$, counted with multiplicity. By Cauchy's inequality we have

$$\left(\frac{\left| \frac{b_{m-1}}{b_m} \right|}{m} \right)^m = \left(\frac{x_1 + \ldots + x_m}{m} \right)^m \geq x_1 x_2 \ldots x_m = \left| \frac{b_0}{b_m} \right|.$$

We can easily see that the last inequality is equivalent to

$$\left(\frac{a_{n-1}}{n} \right)^m \geq \frac{1}{\binom{n}{m}} |a_{n-m}|$$

and it implies

$$\left(\frac{\lambda_1 + \lambda_2 + \ldots + \lambda_n}{n} \right)^m \geq \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < ... < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_m}.$$

Proof of the Claim is complete.

Let us now complete our proof. By using the Claim, we obtain that

$$\frac{1}{n} |\Delta u| \geq \frac{1}{\binom{n}{m}^{\frac{1}{m}}} \left| (dd^c u)^m \wedge \beta^{n-m} \right|^{\frac{1}{m}}$$

for any $u \in sh_m(D) \cap C^2(D)$. It gives us

$$\frac{1}{\binom{n}{m}^{\frac{1}{m}}} \int_D \left|(dd^c u)^m \wedge \beta^{n-m}\right|^{\frac{1}{m}} \leq \frac{1}{n} \int_D \Delta u$$

and thanks to Favard's inequality (see [10]) there exists a constant $C > 0$ (depending on measure of D) such that

$$C \int_D \left|(dd^c u)^m \wedge \beta^{n-m}\right|^{\frac{1}{m}} \geq \sqrt[m]{\int_D (dd^c u)^m \wedge \beta^{n-m}}.$$

Thus, there exists a constant $C > 0$ such that

$$\frac{1}{\binom{n}{m}^{\frac{1}{m}}} \sqrt[m]{\int_D (dd^c u)^m \wedge \beta^{n-m}} \leq \frac{C}{n} \int_D \Delta u. \tag{3}$$

By the smooth approximation $u_j \downarrow u$ and the convergence of currents (see [1]) $(dd^c u_j)^m \wedge \beta^{n-m} \mapsto (dd^c u)^m \wedge \beta^{n-m}$ and $\Delta u_j \mapsto \Delta u$, we obtain (3) for any m-subharmonic function. It completes the proof.

□

From Theorem 1 we have the following corollary.

Corollary 1. *Δ_m-capacity of E is zero if and only if E is a m-polar set.*

Remark 1. *Actually, we can obtain a similar result if we define Δ_m^k-capacity by using $(dd^c u)^k \wedge \beta^{n-k}$, with $1 \leq k \leq m$, instead of $dd^c u \wedge \beta^{n-1}$. All the above properties can be proven by similar technique for Δ_m^k-capacity. However, in this paper, our focus is exclusively on the Laplace operator, which is linear and, therefore, deemed more important.*

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