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IDENTIFICATION OF SOURCES IN A BOUNDARY VALUE PROBLEM FOR BENNEY-LUKE TYPE DIFFERENTIAL EQUATION WITH INTEGRAL CONDITIONS

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Abstract

In three-dimensional domain a problem of identification of recourses for Benney–Luke type partial differential equation of the even order with integral form conditions, spectral parameter and small positive parameters in mixed derivatives is considered. The solution of this partial differential equation is studied in the class of regular functions. The Fourier series method is used. Using this Fourier method, is obtained a countable system of ordinary differential equations. So, the nonlocal boundary value problem is integrated as an ordinary differential equation. When we define the arbitrary integration constants there are possible five cases with respect to the spectral parameter. By the aid of given additional condition, we obtained the presentations with respect to redefinition functions. Using the Cauchy–Schwarz inequality and the Bessel inequality, we proved the absolute and uniform convergence of the obtained Fourier series.

Keywords: *identification of sources, Benney-Luke type differential equation, Fourier method, absolute and convergence, regular solvability.*

Mathematics Subject Classification (2010): *35A02, 35M10, 35S05.*

Introduction

The theory of direct boundary and inverse boundary value problems is currently one of the most important sections of the theory of differential equations. Studies of many problems of gas dynamics, theory of elasticity, theory of plates and shells are described by high-order partial differential equations. Partial differential equations of Boussinesq type and Benney-Luke type have differential applications in different branch of sciences (see, for example, [5, 6, 17]). Therefore, a large number of works are devoted to the study of inverse problems for differential and integro-differential equations (see, for example, [2, 7, 11, 13, 14, 15, 16, 18, 23, 24]). In cases where the boundary of the flow domain of a physical process is unavailable for measurements, nonlocal conditions in integral form can serve as additional information sufficient for unique solvability of the problem [8]. Therefore, in recent years, research on the study of direct and inverse nonlocal boundary value problems for differential and integro-differential equations with integral conditions has been intensified (see, for example, [1, 3, 4, 9, 10, 12, 19, 20, 21, 22], [23]-[30]).

In this paper, we study the regular solvability of a nonlocal inverse boundary value problem for a Benney-Luke type differential equation with spectral parameter and small positive parameters. In studying one-valued solvability and constructing solutions, the presence of spectral parameter plays an important role.

In three-dimensional domain $\Omega = \{(t\,,\,x\,,\,y)\,|\,0 < t < T\,,\;0 < x\,,\,y < l\,\}$ a partial differential equation of the following form is considered

$$D\,[U] = \alpha\,(t)\,\beta\,(x,\,y) \tag{1}$$

with nonlocal conditions on the integral form

$$U\,(T,\,x,\,y) + \int_0^T U\,(t,\,x,\,y)\,d\,t = \varphi_1(x,\,y)\,,\quad 0 \le x\,,\,y \le l\,, \tag{2}$$

$$U_t\,(T,\,x,\,y) + \int_0^T U_t\,(t,\,x,\,y)\,t\,d\,t = \varphi_2\,(x,\,y)\,,\quad 0 \le x,\,y \le l, \tag{3}$$

where T and l are given positive real numbers,

$$D\,[U] = \left[\frac{\partial^2}{\partial\,t^2} - \frac{\partial^2}{\partial\,t^2}\left(\varepsilon_1\frac{\partial^{2k}}{\partial\,x^{2k}} - \varepsilon_2\frac{\partial^{4k}}{\partial\,x^{4k}} + \varepsilon_1\frac{\partial^{2k}}{\partial\,y^{2k}} - \varepsilon_2\frac{\partial^{4k}}{\partial\,y^{4k}}\right) - \right.$$

$$\left. - \,\omega^2\left(\frac{\partial^{2k}}{\partial\,x^{2k}} - \frac{\partial^{4k}}{\partial\,x^{4k}} + \frac{\partial^{2k}}{\partial\,y^{2k}} - \frac{\partial^{4k}}{\partial\,y^{4k}}\right)\right]U\,(t,\,x,\,y),$$

ω is positive spectral parameter, ε_1, ε_2 are positive small parameters, k is given positive integer, $\alpha\,(t\,) \in C\,(\Omega_T)$, $\Omega_T \equiv [0\,;\,T]\,$, $\Omega_l \equiv [0\,;\,l]\,$, $\beta\,(x,\,y) \in C\,(\Omega_l^2)$ is known function, $\varphi_i(x,\,y)\,(i = 1,\,2)$ are redefinition functions, $\Omega_l^2 \equiv \Omega_l \times \Omega_l$, $\Omega_l \equiv [0\,;\,l]\,$. We assume that for given functions are true the following boundary conditions

$$\varphi_i(0,\,y) = \varphi_i(l,\,y) = \varphi_i(x,\,0) = \varphi_i\,(x,\,l) = 0,\;\; i = 1,\,2,$$

$$\beta\,(0,\,y) = \beta\,(l,\,y) = \beta\,(x,\,0) = \beta\,(x,\,l) = 0.$$

Problem Statement. We find the triple of functions $\{U\,(t,\,x,\,y);\;\varphi_1\,(x,\,y),\;\varphi_2\,(x,\,y)\}$, first of which satisfies differential equation (1), nonlocal integral conditions (2) and (3), zero boundary value conditions for $0 \le t \le T$

$$U\,(t,\,0,\,y) = U\,(t,\,l,\,y) = U\,(t,\,x,\,0) = U\,(t,\,x,\,l) =$$

$$= \frac{\partial^2}{\partial\,x^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial\,x^2}U\,(t,\,x,\,l) =$$

$$= \frac{\partial^2}{\partial y^2}U\,(t,\,0,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,l,\,y) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,0) = \frac{\partial^2}{\partial y^2}U\,(t,\,x,\,l) = \ldots =$$

$$= \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,0,\,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,l,\,y) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,x,\,0) = \frac{\partial^{4k-2}}{\partial x^{4k-2}}U\,(t,\,x,\,l) =$$

$$= \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,0,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,l,y) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,x,0) = \frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t,x,l) = 0\,, \tag{4}$$

class of functions

$$U(t,\,x,\,y)\in C(\overline{\Omega})\cap C^{2,\,4k,\,4k}_{t,\,x,\,y}(\Omega)\cap C^{2+4k+0}_{t,\,x,\,y}(\Omega)\cap C^{2+0+4k}_{t,\,x,\,y}(\Omega) \tag{5}$$

and additional conditions

$$U(t_i,\,x,\,y)=\psi_i(x,\,y)\,,\quad i=1,\,2,\;\;0<t_1<t_2<T,\;\;0\le x\,,\,y\le l\,, \tag{6}$$

where $\psi_i(x,\,y)$ are given smooth functions and

$$\psi_i(0,\,y)=\psi_i(l,\,y)=\psi_i(x,\,0)=\psi_i(x,\,l)=0,$$

$C^{2+4k+0}_{t,\,x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{2+4k}U(t,x,y)}{\partial t^2\partial x^{4k}}$ on Ω, while $C^{2+0+4k}_{t,\,x,\,y}(\Omega)$ is the class of continuous functions $\frac{\partial^{2+4k}U(t,x,y)}{\partial t^2\partial y^{4k}}$ on Ω, $\overline{\Omega}=\{(t\,,\,x\,,\,y)\,|\;0\le t\le T\,,\;0\le x\,,\,y\le l\,\}$, $\frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t\,,\,x\,,\,l)$ we understand as $\frac{\partial^{4k-2}}{\partial y^{4k-2}}U(t\,,\,x\,,\,y)\Big|_{y=l}$.

1 Expansion of the solution of the problem in a Fourier series for regular values of spectral parameter

Nontrivial solutions of the problem are sought as a Fourier series

$$U(t,\,x,\,y)=\sum_{n,\,m=1}^{\infty}u_{n\,,m}(t)\,\vartheta_{n\,,m}(x,\,y), \tag{7}$$

where

$$u_{n,\,m}(t)=\int_0^l\int_0^l U(t,\,x,\,y)\,\vartheta_{n,\,m}(x,\,y)\;d\,x\,d\,y, \tag{8}$$

$$\vartheta_{n,\,m}(x,\,y)=\frac{2}{l}\,\sin\frac{\pi\,n}{l}\,x\,\sin\frac{\pi\,m}{l}\,y, n,\,m=1\,,\,2\,,\,\ldots$$

We also suppose that the following functions are expand to Fourier series

$$\beta(x,\,y)=\sum_{n,\,m=1}^{\infty}\beta_{n\,,m}\,\vartheta_{n\,,m}(x,\,y), \tag{9}$$

where

$$\beta_{n,\,m}=\int_0^l\int_0^l\beta(x,\,y)\,\vartheta_{n,\,m}(x,\,y)\,d\,x\,d\,y. \tag{10}$$

Substituting Fourier series (7) and (9) into partial differential equation (1), we obtain the countable system of ordinary differential equations of second order

$$u''_{n\,,m}(t)+\lambda^{2k}_{n\,,m}\,\omega^2\,u_{n\,,m}(t)=\frac{\alpha(t)\,\beta_{n\,,m}}{1+\mu^{2k}_{n\,,m}\left(\varepsilon_1+\varepsilon_2\mu^{2k}_{n\,,m}\right)}\,, \tag{11}$$

where

$$\lambda_{n,m}^{2k}=\frac{\mu_{n,m}^{2k}\left(1+\mu_{n,m}^{2k}\right)}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)},\mu_{n,m}^{k}=\left(\frac{\pi}{l}\right)^k\sqrt{n^{2k}+m^{2k}}\,.$$

The second order countable system of differential equations (11) is solved by the variation method of arbitrary constants

$$u_{n,m}(t)=A_{1n,m}\cos\left(\lambda_{n,m}^{k}\omega t\right)+A_{2n,m}\sin\left(\lambda_{n,m}^{k}\omega t\right)+\gamma_{n,m}(t), \tag{12}$$

where $\gamma_{n,m}(t)=\frac{\beta_{n,m}}{\lambda_{n,m}^{k}\omega}h_{n,m}(t)$, $A_{1,n,m}$ and $A_{2,n,m}$ are arbitrary constants,

$$h_{n,m}(t)=\frac{1}{1+\mu_{n,m}^{2k}\left(\varepsilon_1+\varepsilon_2\mu_{n,m}^{2k}\right)}\int_0^t\sin\left(\lambda_{n,m}^{k}\omega(t-s)\right)\alpha(s)\,ds.$$

Using Fourier coefficients (8), the integral conditions (2) and (3) are written in the following form

$$u_{n,m}(T)+\int_0^T u_{n,m}(t)\,dt=$$

$$=\int_0^l\int_0^l\left(U(T,x,y)+\int_0^T U(t,x,y)\,dt\right)\vartheta_{n,m}(x,y)\,dx\,dy=$$

$$=\int_0^l\int_0^l\varphi_1(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy=\varphi_{1n,m}, \tag{13}$$

$$u'_{n,m}(T)+\int_0^T u'_{n,m}(t)\,t\,dt=$$

$$=\int_0^l\int_0^l\left(U_t(T,x,y)+\int_0^T U_t(t,x,y)\,t\,dt\right)\vartheta_{n,m}(x,y)\,dx\,dy=$$

$$=\int_0^l\int_0^l\varphi_2(x,y)\,\vartheta_{n,m}(x,y)\,dx\,dy=\varphi_{2n,m}. \tag{14}$$

To find the unknown coefficients $A_{1n,m}$ and $A_{2n,m}$ in (12), we use conditions (13) and (14) and obtain the system

$$\begin{cases} A_{1n,m}\,\sigma_{1n,m}(\omega)+A_{2n,m}\sigma_{2n,m}(\omega)=\varphi_{01n,m}, \\ A_{1n,m}\,\sigma_{3n,m}(\omega)+A_{2n,m}\sigma_{4n,m}(\omega)=\varphi_{02n,m}, \end{cases} \tag{15}$$

where

$$\sigma_{1n,m}(\omega)=\frac{\lambda_{n,m}^{k}\omega\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\sin\left(2\lambda_{n,m}^{k}\omega T\right)}{\lambda_{n,m}^{k}\omega}$$

$$\sigma_{2n,m}(\omega)=\frac{-\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\lambda_{n,m}^{k}\omega\sin\left(2\lambda_{n,m}^{k}\omega T\right)+1}{\lambda_{n,m}^{k}\omega},$$

$$\sigma_{3n,m}(\omega)=\frac{-\lambda_{n,m}^{k}\omega T\cos\left(2\lambda_{n,m}^{k}\omega T\right)-\lambda_{n,m}^{k}\omega T+\left[1+\left(\lambda_{n,m}^{k}\omega\right)^2\right]\sin\left(2\lambda_{n,m}^{k}\omega T\right)}{\left(\lambda_{n,m}^{k}\omega\right)^2},$$

$$\sigma_{4n,m}(\omega)=\frac{\left[1+\left(\lambda_{n,m}^{k}\omega\right)^{2}\right]\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\lambda_{n,m}^{k}\omega T\sin\left(2\lambda_{n,m}^{k}\omega T\right)-1}{\left(\lambda_{n,m}^{k}\omega\right)^{2}},$$

$$\varphi_{01n,m}=\varphi_{1n,m}-\left(\gamma_{n,m}(T)+\int_0^T\gamma_{n,m}(t)\,dt\right),$$

$$\varphi_{02n,m}=\varphi_{2n,m}-\left(\gamma'_{n,m}(T)+\int_0^T\gamma'_{n,m}(t)\,t\,dt\right).$$

The system (15) unique solvable, if the following determinant is nonzero

$$\sigma_{5n,m}(\omega)=\sigma_{1n,m}(\omega)\cdot\sigma_{4n,m}(\omega)-\sigma_{2n,m}(\omega)\cdot\sigma_{3n,m}(\omega)\neq 0.$$

To uniquely determine $A_{1n,m}$ and $A_{2n,m}$ from system (15), we calculate the values of the spectral parameter ω presented in the coefficients $\sigma_{in,m}(\omega)$, $i=\overline{1,4}$. The coefficients $\sigma_{in,m}(\omega)$, $i=\overline{1,4}$ can go to zero for some values of the parameter ω from the positive semi-axis $(0;\ \infty)$.

1. We assume that $\sigma_{1n,m}(\omega)=0$. Then we obtain

$$\lambda_{n,m}^{k}\omega\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\sin\left(2\lambda_{n,m}^{k}\omega T\right)=0.$$

Hence, we have a trigonometric equation $\tan\left(2\lambda_{n,m}^{k}\omega T\right)=-\lambda_{n,m}^{k}\omega$ with respect to parameter ω. So, the condition $\sigma_{1n,m}(\omega)=0$ and a trigonometric equation $\tan\left(2\lambda_{n,m}^{k}\omega T\right)=-\lambda_{n,m}^{k}\omega$ are equivalent.

2. Let $\sigma_{2n,m}(\omega)=0$. Then we have a trigonometric equation

$$\cos\left(2\lambda_{n,m}^{k}\omega T\right)-\lambda_{n,m}^{k}\omega\sin\left(2\lambda_{n,m}^{k}\omega T\right)=1.$$

3. When $\sigma_{3n,m}(\omega)=0$, we come to the following trigonometric equation

$$-\lambda_{n,m}^{k}\omega T\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\left[1+\left(\lambda_{n,m}^{k}\omega\right)^{2}\right]\sin\left(2\lambda_{n,m}^{k}\omega T\right)=\lambda_{n,m}^{k}\omega T.$$

4. If we put $\sigma_{4n,m}(\omega)=0$, then have a trigonometric equation

$$\left[1+\left(\lambda_{n,m}^{k}\omega\right)^{2}\right]\cos\left(2\lambda_{n,m}^{k}\omega T\right)+\lambda_{n,m}^{k}\omega T\sin\left(2\lambda_{n,m}^{k}\omega T\right)-1.$$

The set of all values of the spectral parameter ω, consisting of positive solutions of trigonometric equations $\sigma_{in,m}(\omega)=0$, we denote by Λ_i, respectively, $i=\overline{1,4}$. It easy to prove, that $\Lambda_i\cap\Lambda_j=\emptyset$, $i,j=\overline{1,4}$, $i\neq j$. For example, we show that $\Lambda_1\cap\Lambda_2=\emptyset$. In this order we assume, that $\sigma_{1n,m}(\omega)=\sigma_{2n,m}(\omega)=0$. By virtue of our assumption, we have $\sigma_{1n,m}^{2}(\omega)+\sigma_{2n,m}^{2}(\omega)=0$. Hence, we obtain a trigonometric equation

$$\frac{1}{\sqrt{1+\left(\lambda_{n,m}^{k}\omega\right)^{2}}}\cos\left(2\lambda_{n,m}^{k}\omega T\right)-\frac{\lambda_{n,m}^{k}\omega}{\sqrt{1+\left(\lambda_{n,m}^{k}\omega\right)^{2}}}\sin\left(2\lambda_{n,m}^{k}\omega T\right)=\frac{\left(\lambda_{n,m}^{k}\omega\right)^{2}+2}{2\sqrt{1+\left(\lambda_{n,m}^{k}\omega\right)^{2}}}$$

and this equation should have a solution. As you know, this trigonometric equation is solvable, if right-hand side of this equation lays in the interval $[-1; 1]$. But, we show that $\frac{\left(\lambda_{n,m}^{k}\omega\right)^2+2}{2\sqrt{1+\left(\lambda_{n,m}^{k}\omega\right)^2}} > 1$. Indeed, $2+\lambda_{n,m}^{k}\omega^2 > 2\sqrt{1+\lambda_{n,m}^{k}\omega^2}$. Both sides of the inequality are greater than one. Therefore, they can be squared both sides:

$$\left(2+\lambda_{n,m}^{k}\omega^2\right)^2 > 4\left(1+\lambda_{n,m}^{k}\omega^2\right) \Rightarrow 4+4\,\lambda_{n,m}^{k}\omega^2+\left(\lambda_{n,m}^{k}\omega^2\right)^2 > 4+4\,\lambda_{n,m}^{k}\omega^2$$

or $\left(\lambda_{n,m}^{2}\omega^2\right)^2 > 0$. Hence, we obtain that $\frac{\left(\lambda_{n,m}^{k}\omega\right)^2+2}{2\sqrt{1+\left(\lambda_{n,m}^{k}\omega\right)^2}} > 1$. So, this trigonometric equation has no any solution. Therefore, our assumption $\sigma_{1\,n,m}(\omega)=\sigma_{2\,n,m}(\omega)=0$is not true. In other cases, this statement is proved similarly. Consequently, there are values of the parameter ω, for which $\sigma_{5\,n,m}(\omega)\neq 0$. We introduce the denotation $\Lambda_5 = (0;\infty)\setminus\left(\bigcup_{j=1}^{4}\Lambda_j\right)$. It is possible there five cases: 1) $\sigma_{1\,n,m}(\omega)=0$; 2) $\sigma_{2\,n,m}(\omega)=0$; 3) $\sigma_{3\,n,m}(\omega)=0$; 4) $\sigma_{4\,n,m}(\omega)=0$; 5) $\sigma_{j\,n,m}(\omega)\neq 0\,,\quad j=\overline{1,\,4}$.

Solve the system of algebraic equations (15). Then from presentation (12) we derived that

$$u_{n,m}(t)=\varphi_{1\,n,m}B_{j\,n,m}(t)+\varphi_{2\,n,m}C_{j\,n,m}(t)+\frac{\beta_{n,m}}{\lambda_{n,m}^{k}\,\omega}E_{j\,n,m}(t)\,,\ \ \omega\in\Lambda_j\,,\ j=\overline{1,\,5}\,, \tag{16}$$

where Fourier coefficients $\beta_{n\,,m}$ are defined by the presentations (10),

$$E_{j\,n,m}(t)=h_{n\,,m}(t)-B_{j\,n,m}(t)\ \left[\int_0^T h_{n,m}(t)\,d\,t+h_{n,m}(T)\right]-$$

$$-C_{j\,n,m}(t)\ \left[\int_0^T h'_{n,m}(t)\,t\,d\,t+h'_{n,m}(T)\right],$$

$$B_{1\,n,m}(t)=\frac{\sin\left(\lambda_{n\,,m}^{k}\omega\,t\right)}{\sigma_{2\,n,m}(\omega)}-\frac{\sigma_{4\,n,m}(\omega)}{\sigma_{2\,n,m}(\omega)}\frac{\cos\left(\lambda_{n\,,m}^{k}\omega\,t\right)}{\sigma_{3\,n,m}(\omega)},\quad C_{1\,n,m}(t)=\frac{\cos\left(\lambda_{n\,,m}^{k}\omega\,t\right)}{\sigma_{3\,n,m}(\omega)},$$

$$B_{2\,n,m}(t)=\frac{\cos\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{1\,n,m}(\omega)}-\frac{\sigma_{3\,n,m}(\omega)}{\sigma_{1\,n,m}(\omega)}\frac{\sin\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{4\,n,m}(\omega)},\quad C_{2\,n,m}(t)=\frac{\sin\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{4\,n,m}(\omega)},$$

$$B_{3\,n,m}(t)=\frac{\cos\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{1\,n,m}(\omega)},\quad C_{3\,n,m}(t)=\frac{\sin\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{4\,n,m}(\omega)}-\frac{\sigma_{2\,n,m}(\omega)}{\sigma_{1\,n,m}(\omega)}\frac{\cos\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{4\,n,m}(\omega)},$$

$$B_{4\,n,m}(t)=\frac{\sin\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{2\,,n\,,m}(\omega)},\quad C_{4\,n,m}(t)=\frac{\cos\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{3\,n,m}(\omega)}-\frac{\sigma_{1\,n,m}(\omega)}{\sigma_{2\,n,m}(\omega)}\frac{\sin\left(\lambda_{n,m}^{k}\omega\,t\right)}{\sigma_{3\,n,m}(\omega)},$$

$$B_{5\,n,m}(t)=\frac{1}{\sigma_{5\,n,m}(\omega)}\left[\sigma_{4\,n,m}(\omega)\ \cos\left(\lambda_{n,m}^{k}\omega\,t\right)-\sigma_{3\,n,m}(\omega)\ \sin\left(\lambda_{n,m}^{k}\omega\,t\right)\right],$$

$$C_{5\,n,m}(t)=\frac{1}{\sigma_{5\,n,m}(\omega)}\left[-\sigma_{2\,n,m}(\omega)\ \cos\left(\lambda_{n,m}^{k}\omega\,t\right)+\sigma_{1\,n,m}(\omega)\ \sin\left(\lambda_{n,m}^{k}\omega\,t\right)\right],$$

$$\sigma_{5\,n,m}(\omega)=\sigma_{1\,n,m}(\omega)\,\sigma_{4\,n,m}(\omega)-\sigma_{2\,n,m}(\omega)\,\sigma_{3\,n,m}(\omega)\neq 0\,,\quad \omega\in\Lambda_5\,.$$

Substituting the presentation of Fourier coefficients (16) of main unknown function into Fourier series (7), for values of parameter $\omega \in \Lambda_j \ \left(j=\overline{1,\,5}\right)$ we obtain

$$U\left(t,\,x,\,y\right)=\sum_{n,\,m=1}^{\infty}\vartheta_{\,n\,,m}\left(x,\,y\right)\times$$

$$\times\left[\varphi_{\,1\,n,\,m}\,B_{j\,n,\,m}\left(t\right)+\varphi_{\,2\,n,\,m}\,C_{j\,n,\,m}\left(t\right)+\beta_{\,n\,,m}\frac{E_{\,j\,n,\,m}\left(t\right)}{\lambda_{\,n\,,m}^{k}\,\omega}\right],\ \ j=\overline{1,\,5}. \tag{17}$$

Fourier series (17) is a formal solution of the direct problem (1)-(5).

2 Redefinition functions

Using the additional conditions (6) and taking into account (9) and (10), for regular values of parameter $\omega \in \Lambda_j \ \left(j=\overline{1,\,5}\right)$ we obtain from Fourier series (17) following linear system of countable system for Fourier coefficients of redefinition functions

$$\varphi_{\,1n,\,m}\,B_{j\,n,\,m}(t_i)+\varphi_{\,2\,n,\,m}\,C_{j\,n,\,m}(t_i)=\tau_{i\,j\,n,\,m},\ \ i=1,\,2, \tag{18}$$

where

$$\tau_{i\,j\,n,\,m}=\psi_{i\,n,\,m}-\beta_{n,\,m}\frac{E_{\,j\,n,\,m}\left(t_i\right)}{\lambda_{\,n\,,m}^{k}\,\omega},\ \ \ j=\overline{1,\,5}, \tag{19}$$

$$\psi_{\,i\,n,\,m}\ =\ \int_0^l\int_0^l\psi_i(x,\,y)\,\vartheta_{\,n,\,m}\left(x,\,y\right)d\,x\,d\,y,\quad i=1,\,2. \tag{20}$$

The system of algebraic equations is solvable, if following determinant is nonzero:

$$\Delta_{\,jn,\,m}\ =\left|\begin{array}{cc} B_{j\,n,\,m}(t_1) & C_{j\,n,\,m}(t_1) \\ B_{j\,n,\,m}(t_2) & C_{j\,n,\,m}(t_2)\end{array}\right|\neq 0. \tag{21}$$

If the nonzero condition (21) is fulfilled, then from the system (18) we obtain

$$\varphi_{\,1n,\,m}\ =\frac{1}{\Delta_{\,jn,\,m}}\left[\tau_{2n,\,m}\,B_{j\,n,\,m}(t_1)-\tau_{1n,\,m}\,B_{j\,n,\,m}(t_2)\right],$$

$$\varphi_{\,2n,\,m}\ =\frac{1}{\Delta_{\,jn,\,m}}\left[\tau_{1n,\,m}\,C_{j\,n,\,m}(t_2)-\tau_{2n,\,m}\,B_{j\,n,\,m}(t_1)\right],\ \ j=\overline{1,\,5}$$

or substituting (19) into last relations, we obtain the following presentations

$$\varphi_{\,i\,n,\,m}\ =\psi_{\,1n,\,m}\,\chi_{\,i\,1n,\,m}\ +\psi_{\,2n,\,m}\,\chi_{\,i\,2n,\,m}\ +\beta_{\,n,\,m}\,\chi_{\,i\,3n,\,m}\,,\ \ i=1,\,2, \tag{22}$$

where

$$\chi_{\,1\,1n,\,m}\ =-\frac{1}{\Delta_{\,jn,\,m}}B_{j\,n,\,m}(t_2),\quad \chi_{\,1\,2n,\,m}\ =\frac{1}{\Delta_{\,jn,\,m}}B_{j\,n,\,m}(t_1),$$

$$\chi_{\,1\,3n,\,m}\ =\frac{1}{\lambda_{\,n\,,m}^{k}\,\omega\,\Delta_{\,jn,\,m}}\left[B_{j\,n,\,m}(t_2)\,E_{\,j\,n,\,m}\left(t_1\right)-B_{j\,n,\,m}(t_1)\,E_{\,j\,n,\,m}\left(t_2\right)\right],$$

$$\chi_{\,2\,1n,\,m}\ =\frac{1}{\Delta_{\,jn,\,m}}C_{j\,n,\,m}(t_2),\quad \chi_{\,2\,2n,\,m}\ =-\frac{1}{\Delta_{\,jn,\,m}}C_{j\,n,\,m}(t_1),$$

$$\chi_{23n,m} = \frac{1}{\lambda_{n,m}^k \omega \Delta_{jn,m}} \left[C_{jn,m}(t_1) E_{jn,m}(t_2) - C_{jn,m}(t_2) E_{jn,m}(t_1)\right], \ j = \overline{1, 5}.$$

Since $\varphi_{in,m}$ are Fourier coefficients (see (13) and (14)), presentations (22) we substitute into Fourier series

$$\varphi_i(x, y) = \sum_{n,m=1}^{\infty} \varphi_{in,m} \vartheta_{n,m}(x, y), \ \ i = 1, 2$$

and obtain

$$\varphi_i(x, y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y) \left[\psi_{i1n,m} \chi_{i1n,m} + \psi_{2n,m} \chi_{i2n,m} + \beta_{n,m} \chi_{i3n,m}\right]. \tag{23}$$

We prove absolutely and uniformly convergence of Fourier series (23) in the domain $\overline{\Omega}$. We use the concepts of the following well-known Banach spaces. Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ with norm

$$\|\varphi\|_{\ell_2} = \sqrt{\sum_{n,m=1}^{\infty} |\varphi_{n,m}|^2} < \infty .$$

The space $L_2(\Omega_l^2)$ of square-summable functions on the domain $\Omega_l^2 = \Omega_l \times \Omega_l$ with norm

$$\|\vartheta(x, y)\|_{L_2(\Omega_l^2)} = \sqrt{\int_0^l \int_0^l |\vartheta(x, y)|^2 dx\, dy} < \infty .$$

Conditions of smoothness. Let for functions

$$\psi_i(x, y), \ \beta(x, y) \in C^{4k}(\Omega_l^2), \ i = 1, 2$$

in the domain Ω_l^2 there exist piecewise continuous $4k+1$ order derivatives.

Then by integrating in parts the functions (10) and (20) $4k+1$ times over every variable E, y, we obtain following relations [23]

$$|\psi_{in,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\psi_{in,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \ \ i = 1, 2, |\beta_{n,m}| = \left(\frac{l}{\pi}\right)^{8k+2} \frac{\left|\beta_{n,m}^{(8k+2)}\right|}{n^{4k+1} m^{4k+1}}, \tag{24}$$

$$\left\|\psi_{in,m}^{(8k+2)}\right\|_{\ell_2} \le \frac{2}{l} \left\|\frac{\partial^{8k+2} \psi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \ \ i = 1, 2, \tag{25}$$

$$\left\|\beta_{n,m}^{(8k+2)}\right\|_{\ell_2} \le \frac{2}{l} \left\|\frac{\partial^{8k+2} \beta(x, y)}{\partial x^{4k+1} \partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}, \tag{26}$$

where

$$\psi_{in,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} \psi_i(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \vartheta_{n,m}(x, y)\, dx\, dy, \ \ i = 1, 2,$$

$$\beta_{n,m}^{(8k+2)} = \int_0^l \int_0^l \frac{\partial^{8k+2} \beta(x, y)}{\partial x^{4k+1} \partial y^{4k+1}} \vartheta_{n,m}(x, y)\, dx\, dy.$$

For regular values of parameter $\omega \in \Lambda_j$ $\left(j = \overline{1, 5}\right)$ we prove that there holds

Theorem 1. Suppose that the conditions of smoothness, nonzero condition (21) and following condition are fulfilled:

$$\sigma_{0i} = \max_{n,m} \{|\chi_{1in,m}|; |\chi_{2in,m}|; |\chi_{3in,m}|\} < \infty, \quad i = 1, 2. \tag{27}$$

Then Fourier series (23) convergence absolutely and uniformly for regular spectral values from the numerical set $\omega \in \Lambda_j$ for each $j = \overline{1, 5}$ and all possible n and m.

Proof. We use formulas (24)-(26) and estimate (27). Using the Cauchy-Schwartz inequality for series (23), we obtain the estimates

$$|\varphi_i(x, y)| \le \sum_{n,m=1}^{\infty} |\vartheta_{n,m}(x, y)| \cdot |\psi_{1n,m}\chi_{1in,m} + \psi_{2n,m}\chi_{2in,m} + \beta_{2n,m}\chi_{3in,m}| \le$$

$$\le \frac{2}{l}\sigma_i \left[\sum_{n,m=1}^{\infty} |\psi_{1n,m}| + \sum_{n,m=1}^{\infty} |\psi_{2n,m}| + \sum_{n,m=1}^{\infty} |\beta_{n,m}|\right] \le$$

$$\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_i \left[\sum_{n,m=1}^{\infty} \frac{\left|\psi_{1n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left|\psi_{2n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}} + \sum_{n,m=1}^{\infty} \frac{\left|\beta_{n,m}^{(8k+2)}\right|}{n^{4k+1}m^{4k+1}}\right] \le$$

$$\le \frac{2}{l}\left(\frac{l}{\pi}\right)^{8k+2} \sigma_i C_{01} \left[\left\|\psi_{1n,m}^{(8k+2)}\right\|_{\ell_2} + \left\|\psi_{2n,m}^{(8k+2)}\right\|_{\ell_2} + \left\|\beta_{n,m}^{(8k+2)}\right\|_{\ell_2}\right] \le$$

$$\le \gamma_{1i}\left[\left\|\frac{\partial^{8k+2}\psi_1(x, y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)} + \left\|\frac{\partial^{8k+2}\psi_2(x, y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)} + \left\|\frac{\partial^{8k+2}\beta(x, y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right] < \infty, \tag{28}$$

where

$$\gamma_{1i} = \sigma_i C_{01}\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{8k+2}, \quad i = 1, 2, \quad C_{01} = \sqrt{\sum_{n,m=1}^{\infty} \frac{1}{n^{8k+2}m^{8k+2}}} < \infty.$$

From estimate (28) implies the absolutely and uniformly convergence of Fourier series (23). The theorem 1 is proved.

3 Main unknown function

We determined the redefinition functions as a Fourier series (23). So, redefinition functions are known. Using representations (22), Fourier series (17), the main unknown function we can present as

$$U(t, x, y) = \sum_{n,m=1}^{\infty} \vartheta_{n,m}(x, y)\left[\psi_{1n,m}W_{1jn,m}(t) + \psi_{2n,m}W_{2jn,m}(t) + \beta_{n,m}W_{3jn,m}(t)\right], \tag{29}$$

where

$$W_{ijn,m}(t) = \chi_{1in,m}B_{jn,m}(t) + \chi_{2in,m}C_{jn,m}(t), \quad i = 1, 2,$$

$$W_{3jn,m}(t) = \chi_{13n,m}B_{jn,m}(t) + \chi_{23n,m}C_{jn,m}(t) + \frac{E_{jn,m}(t)}{\lambda_{n,m}^k\omega}, \quad j = \overline{1, 5}.$$

To establish the uniqueness of the function $U\,(t,\,x,\,y)$ we suppose that there are two functions U_1 and U_2 satisfying the given conditions (1)-(6). Then their difference $U = U_1 - U_2$ is a solution of differential equation (1), satisfying conditions (2)-(6) with functions $\psi_i(x,\,y) \equiv 0$ $(i = 1,\,2)$. By virtue of relations (10) and (20) we have $\psi_{i\,n,\,m} = 0$ $(i = 1,\,2)$. Hence, we obtain from formulas (8) and (29) in the domain Ω, that there follows the following zero identity

$$\int_0^l \int_0^l U\,(t,\,x,\,y)\,\vartheta_{\,n,\,m}\,(x,\,y)\;d\,x\,d\,y \equiv 0.$$

Hence, by virtue of the completeness of the systems of eigenfunctions $\left\{\sqrt{\frac{2}{l}}\sin\,\frac{\pi\,n}{l}\,x\right\}$, $\left\{\sqrt{\frac{2}{l}}\sin\,\frac{\pi\,m}{l}\,y\right\}$ in $L_2\,\left(\Omega_l^2\right)$ we deduce that $U\,(t,\,x,\,y) \equiv 0$ for all $x\,\in \Omega_l^2 \equiv [0\,,\,l]^2$ and $t \in \Omega_T \equiv [0\,;\,T]\,.$
Therefore, for regular values of spectral parameter ω the function $U\,(t,\,x,\,y)$ is unique solution of differential equation (1) with conditions (2)–(6), if this function exists in the domain $\Omega\,.$

Theorem 2. Let the conditions of the theorem 1 be fulfilled. Then for regular values of spectral parameter $\omega \in \Lambda_j\,\left(j = \overline{1,\,5}\right)$ the series (29) converge. At the same time, their term by term differentiation is possible.

Proof. By virtue of conditions of the theorem 1, the functions $W_{i\,j\,n,\,m}\,(t)$, $i = \overline{1,\,3}$, $j = \overline{1,\,5}$ uniformly bounded on the segment $[0;\,T]$. So for any positive integers $n,\,m$ there exist finite constant C_2, that there takes place the following estimates

$$\max_{n,\,m}\left\{\max_{i=\overline{1,\,3}}\max_{j=\overline{1,\,5}}\max_{0\le t\le T}|\,W_{ij\,n,\,m}\,(t)\,|\,;\,\max_{i=\overline{1,\,3}}\max_{j=\overline{1,\,5}}\max_{0\le t\le T}\left|\,W''_{ij\,n,\,m}\,(t)\,\right|\right\} \le C_{0\,2}\,. \tag{30}$$

Using estimates (24)-(26) and (30), analogously to the estimate (28), for series (29) we obtain

$$|\,U\,(t,\,x,\,y)\,| \le \sum_{n,\,m=1}^{\infty}|\,\vartheta_{\,n,\,m}\,(x,\,y)\,|\,\cdot\,|\,\psi_{\,1\,n,\,m}\,W_{1j\,n,\,m}\,(t)+$$

$$+\psi_{\,2\,n,\,m}\,W_{2j\,n,\,m}\,(t) + \beta_{\,n,\,m}\,W_{3\,j\,n,\,m}\,(t)\,| \le$$

$$\le \gamma_2\left[\left\|\frac{\partial^{8k+2}\psi_1\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_2\left(\Omega_l^2\right)} + \left\|\frac{\partial^{8k+2}\psi_2\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_2\left(\Omega_l^2\right)} + \left\|\frac{\partial^{\,8k+2}\beta\,(x,\,y)}{\partial\,x^{4k+1}\partial\,y^{4k+1}}\right\|_{L_2\left(\Omega_l^2\right)}\right] < \infty, \tag{31}$$

where $\gamma_2 = C_{01}\,C_{02}\,\left(\frac{2}{l}\right)^2\,\left(\frac{l}{\pi}\right)^{8k+2}$.
Function (29) differentiate the required number of times

$$U_{tt}(t,\,x,\,y) = \sum_{n,\,m=1}^{\infty}\vartheta_{\,n\,,\,m}\,(x,\,y)\,\left[\psi_{\,1\,n,\,m}\,W''_{1j\,n,\,m}\,(t) + \psi_{\,2\,n,\,m}\,W''_{2j\,n,\,m}\,(t) + \beta_{\,n,\,m}\,W''_{3j\,n,\,m}\,(t)\right], \tag{32}$$

$$\frac{\partial^{\,4k}}{\partial\,x^{\,4k}}U\,(t,\,x,\,y) = \sum_{n,\,m=1}^{\infty}\left(\frac{\pi\,n}{l}\right)^{4k}\vartheta_{\,n\,,\,m}\,(x,\,y)\times$$

$$\times\left[\psi_{\,1\,n,\,m}\,W_{1j\,n,\,m}\,(t) + \psi_{\,2\,n,\,m}\,W_{2j\,n,\,m}\,(t) + \beta_{\,n,\,m}\,W_{3j\,n,\,m}\,(t)\right], \tag{33}$$

$$\frac{\partial^{4k}}{\partial y^{4k}}U(t,x,y)=\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}\vartheta_{n,m}(x,y)\times$$

$$\times\left[\psi_{1n,m}W_{1jn,m}(t)+\psi_{2n,m}W_{2jn,m}(t)+\beta_{n,m}W_{3jn,m}(t)\right]. \tag{34}$$

The expansions of the following functions into Fourier series are defined in the domain Ω_l^2 in a similar way

$$\frac{\partial^{4k+2}}{\partial t^2\partial x^{4k}}U(t,x,y),\ \frac{\partial^{4k+2}}{\partial t^2\partial y^{4k}}U(t,x,y).$$

The convergence of series (32) is proved similarly to the proof of the convergence of series (29). Let us show the convergence of series (33) and (34). Taking into account formulas (24)-(26) and (30) and applying the Cauchy-Schwarz inequality, we obtain:

$$\left|\frac{\partial^{4k}}{\partial x^{4k}}U(t,x,y)\right|\leq\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}|u_{n,m}(t)|\cdot|\vartheta_{n,m}(x,y)|\leq$$

$$\leq\sum_{n,m=1}^{\infty}\left(\frac{\pi n}{l}\right)^{4k}|\vartheta_{n,m}(x,y)|\cdot|\psi_{1n,m}W_{1jn,m}(t)+\psi_{2n,m}W_{2jn,m}(t)+\beta_{2n,m}W_{3jn,m}(t)|\leq$$

$$\leq\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}C_{02}\left[\sum_{n,m=1}^{\infty}n^{4k}|\psi_{1n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|\psi_{2n,m}|+\sum_{n,m=1}^{\infty}n^{4k}|\beta_{n,m}|\right]\leq$$

$$\leq\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}C_{02}\left[\sum_{n,m=1}^{\infty}\frac{\left|\psi_{1n,m}^{(8k+2)}\right|}{nm^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|\psi_{2n,m}^{(8k+2)}\right|}{nm^{4k+1}}+\sum_{n,m=1}^{\infty}\frac{\left|\beta_{n,m}^{(8k+2)}\right|}{nm^{4k+1}}\right]\leq$$

$$\leq\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}C_{02}C_{03}\left[\left\|\psi_{1n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\psi_{2n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\beta_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq\gamma_3\left[\left\|\frac{\partial^{8k+2}\psi_1(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}\psi_2(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}+\left\|\frac{\partial^{8k+2}\beta(x,y)}{\partial x^{4k+1}\partial y^{4k+1}}\right\|_{L_2(\Omega_l^2)}\right]<\infty, \tag{35}$$

where $\gamma_3=\left(\frac{2}{l}\right)^2\left(\frac{l}{\pi}\right)^{4k+2}C_{02}C_{03},\ C_{03}=\sqrt{\sum_{n,m=1}^{\infty}\frac{1}{nm^{8k+2}}}<\infty;$

$$\left|\frac{\partial^{4k}}{\partial y^{4k}}U(t,x,y)\right|\leq\sum_{n,m=1}^{\infty}\left(\frac{\pi m}{l}\right)^{4k}|u_{n,m}(t)|\cdot|\vartheta_{n,m}(x,y)|\leq$$

$$\leq\frac{2}{l}\left(\frac{\pi}{l}\right)^{4k}C_{02}\left[\sum_{n,m=1}^{\infty}m^{4k}|\psi_{1n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|\psi_{2n,m}|+\sum_{n,m=1}^{\infty}m^{4k}|\beta_{n,m}|\right]\leq$$

$$\leq\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}C_{02}\left[\sum_{n,m=1}^{\infty}\frac{\left|\psi_{1n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|\psi_{2n,m}^{(8k+2)}\right|}{n^{4k+1}m}+\sum_{n,m=1}^{\infty}\frac{\left|\beta_{n,m}^{(8k+2)}\right|}{n^{4k+1}m}\right]\leq$$

$$\leq\frac{2}{l}\left(\frac{l}{\pi}\right)^{4k+2}C_{02}C_{04}\left[\left\|\psi_{1n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\psi_{2n,m}^{(8k+2)}\right\|_{\ell_2}+\left\|\beta_{n,m}^{(8k+2)}\right\|_{\ell_2}\right]\leq$$

$$\leq \gamma_4 \left[\left\| \frac{\partial^{8k+2}\psi_1(x,\, y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2}\psi_2(x,\, y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} + \left\| \frac{\partial^{8k+2}\beta(x,\, y)}{\partial x^{4k+1}\partial y^{4k+1}} \right\|_{L_2(\Omega_l^2)} \right] < \infty, \tag{36}$$

where $\gamma_4 = \left(\frac{2}{l}\right)^2 \left(\frac{l}{\pi}\right)^{4k+2} C_{02}C_{04}$, $C_{04} = \sqrt{\sum_{n,\,m=1}^{\infty} \frac{1}{n^{8k+2}m}} < \infty$.

The convergence of Fourier series for functions

$$\frac{\partial^{4k+2}}{\partial t^2 \partial x^{4k}} U(t,\, x,\, y),\ \frac{\partial^{4k+2}}{\partial t^2 \partial y^{4k}} U(t,\, x,\, y)$$

is easy to prove, and the necessary estimates are obtained in a similar way as was done for the cases of estimates (31), (35), (36) in the domain Ω_l^m. Therefore, the function $U(t,\, x,\, y)$ belongs to the class of functions (5). Theorem 2 is proved.

4 Stability of the solution $U(t,\, x,\, y)$ with respect to functions $\psi_1(x,\, y)$, $\psi_2(x,\, y)$

Theorem 3. Suppose that all the conditions of theorem 2 are fulfilled. Then, the function $U(t,\, x,\, y)$ as a solution of the problem (1)–(6) for regular values of the spectral parameter $\omega \in \Lambda_j$ $\left(j = \overline{1,\, 5}\right)$ is stable with respect to given integral function$\psi_1(x,\, y)$, $\psi_2(x,\, y)$.

Proof. We show that the solution of the mixed differential equation (1) $U(t,\, x,\, y)$ is stable with respect to a given functions $\psi_1(x,\, y)$, $\psi_2(x,\, y)$. Let $U_1(t,\, x,\, y)$ and $U_2(t,\, x,\, y)$ be two different solutions of the inverse boundary value problem (1)-(6), corresponding to two different values of the function $\psi_{11}(x)$, $\psi_{12}(x)$ and $\psi_{21}(x)$, $\psi_{22}(x)$, respectively.

We put that max $\{|\psi_{11\,n,m} - \psi_{12\,n,m}|;\, |\psi_{21\,n,m} - \psi_{22\,n,m}|\} < \delta_{n,m}$,where $0 < \delta_{n,m}$ is sufficiently small positive quantity and the series $\sum_{n,m=1}^{\infty} |\delta_{n,m}|$ is convergent. Then, taking this fact into account, by virtue of the conditions of the theorem, from the Fourier series (29) it is easy to obtain that

$$\| U_1(t,\, x,\, y) - U_2(t,\, x,\, y) \|_{C(\overline{\Omega})} \leq \frac{2}{l} C_{02} \sum_{n,\,m=1}^{\infty} [|\psi_{11\,n,m} - \psi_{12\,n,m}| + |\psi_{21\,n,m} - \psi_{22\,n,m}|] <$$

$$< \frac{2}{l} C_{02} \sum_{n,\,m=1}^{\infty} |\delta_{n,m}| < \infty.$$

If we put $\varepsilon = \frac{2}{l} C_{02} \sum_{n,m=1}^{\infty} |\delta_{n,m}| < \infty$, then from last estimate we finally obtain assertions about the stability of the solution of differential equation (1) with respect to a given functions $\psi_1(x,\, y)$, $\psi_2(x,\, y)$. The theorem 3 is proved.

Remark. Analogously one can prove that the function $U(t,\, x,\, y)$ as a solution of the problem (1)–(6) for regular values of the spectral parameter $\omega \in \Lambda_j$ $\left(j = \overline{1,\, 5}\right)$ is stable with respect to given small parameters ε_1, ε_2.

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