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ERROR ESTIMATION FOR THE THIRD-ORDER ACCURACY APPROXIMATE SOLUTION OF THE CAUCHY PROBLEM BY THE TAYLOR FORMULA

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Abstract

A method based on the Taylor formula for the approximate solution of the Cauchy problem for the ordinary differential equation is studied. The problem of estimating the accuracy of the approximate solution generated by this method is considered, and an estimate of high accuracy is obtained for the difference of the exact and approximate solution. Here, different from the known values, an exact expression for the estimation coefficient is found.

Keywords: *the Cauchy problem, approximate solution, the Taylor formula, solution accuracy, error estimate, variable step.*

Mathematics Subject Classification (2010): *65L05.*

Introduction

We study a problem of approximate solving at the Cauchy problem

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{1}$$

that is important in applications of Mathematics.

It is assumed that $f : D \to \mathbb{R}$, D is a convex open set, and the function $f(x,y)$ has the third order continuous partial derivates. If we denote the exact solution of problem (1) by $y_*(x)$, and the approximate solution by $\widetilde{y}(x)$ $(x_0 \leq x \leq x_*)$, the accuracy of the approximate solution is estimated by the following quantity:

$$\Delta = \sup_{x_0 \leq x \leq x_*} \left| y_*(x) - \widetilde{y}(x) \right|. \tag{2}$$

There are many monographs, devoted to this topic, where the estimate of a form

$$\Delta \leq C e^{L(x-x_0)} h^s \tag{3}$$

is derived [1, 2, 3, 4, 10].

Finding the coefficient C is an essential issue for both practical and theoretical points of view. This estimate has so far been considered usually basing on heuristic considerations from the perspective of applications of Computational Mathematics. In the present, inequality (3) will be studied from a pure mathematical point of view [5]. In the article [7], it was derived an inequality of the form (3) for the scheme of the degree of accuracy 2 for autonomous systems(besides, the step was constant). In the article [8], the case of $s = 2$ and the step variable was considered for equation (1). In the present paper, we give an exact formula for the coefficient C in (3) when $s = 3$ and the step is variable.

1 Approximate solution scheme based on Taylor's formula

There exists a solution of problem (1) for the continuous function $f(x, y)$, but it is not easy to estimate the interval where the solution is defined. Therefore, we assume that the solution $y_*(x)$ of the Cauchy problem (1) exists on a given interval $[x_0, x_*]$. The values of the solution $y_*(x)$ are searched on the grid $x_0 < x_1 < x_2 < ... < x_N = x_*$ and the step length $h_n = x_n - x_{n-1}$, can be arbitrary. Let $h = \max\limits_{1 \le x \le N} h_n$.

For the exact solution, we use Taylor's formula with accuracy $o(h^3)$:

$$y_*(x_{n-1} + h_n) = y_*(x_{n-1}) + y'_*(x_{n-1})h_n + \frac{1}{2}y''_*(x_{n-1}){h_n}^2 + \frac{1}{6}{y_*}'''(x_{n-1}){h_n}^3 + R_3, \quad (4)$$

where

$$y'_*(x) = f[x, y_*(x)], \quad y''_*(x) = \{f_x + f_y f\}[x, y_*(x)],$$
$$y'''_*(x) = \left\{f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right\}[x, y_*(x)],$$

R_3 is residual term of Taylor's formula. We substitute these expressions in (4) to obtain

$$y_*(x_{n-1} + h_n) = y_*(x_{n-1}) + f[x_{n-1}, y_*(x_{n-1})]h_n + \{f_x + f_y f\}[x_{n-1}, y_*(x_{n-1})]\frac{{h_n}^2}{2} +$$

$$+ \left\{f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right\}[x_{n-1}, y_*(x_{n-1})]\frac{{h_n}^3}{6} + R_3.$$

If the residual term is dropped, the following sequence is obtained:

$$y_{n+1} = y_n + f[x_n, y_n]h_n + \{f_x + f_y f\}[x_n, y_n]\frac{{h_n}^2}{2} +$$

$$+ \left\{f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)f + f_y f_x\right\}[x_n, y_n]\frac{{h_n}^3}{6}. \quad (5)$$

The sequence $\{y_n\}$ is an approximation for the values of the exact solution $y_*(x)$ on the grid. To prove inequality (3), it is convenient to work with the approximate solution defined on the whole interval $[x_0, x_*]$, but not with the above sequence.

Lemma. If

$$\widetilde{y}(x) = y_0 + \int\limits_{x_0}^{x} \{f[\delta_s, \widetilde{y}(\delta_s)] + (s - \delta_s)(f_y f + f_x)[\delta_s, \widetilde{y}(\delta_s)] +$$

$$+ \frac{(s - \delta_s)^2}{2}\left\{f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right\}[\delta_s, \widetilde{y}(\delta_s)]\}ds, \quad (6)$$

then $\widetilde{y}(x) = y_n$, where $\delta_s = \delta(s)$, which is determined as $\delta(s) = x_k$ for the interval $x_k \le s \le x_{k+1}$, $k = 0, 1, ..., N-1$. In order to derive estimate (2), the condition that the function $y_*(x)$ is defined on the interval $[x_0, x_*]$ is not enough but it must also be known that it lies in a set K, $K \subset \mathbb{R}^2$ on this interval. We assume that K is convex and $(x, y_*(x)) \in K$, $(x, \widetilde{y}(x)) \in K$ for all $x \in [x_0, x_*]$.

We the can define the following numbers:

$$M_0 = \max_{(x,y)\in K} |f(x,y)|, \quad M_{10} = \max_{(x,y)\in K} |f_x(x,y)|, \quad M_{01} = \max_{(x,y)\in K} |f_y(x,y)|,$$

$$M_{20} = \max_{(x,y)\in K} |f_{xx}(x,y)|, \quad M_{11} = \max_{(x,y)\in K} |f_{xy}(x,y)|, \quad M_{02} = \max_{(x,y)\in K} |f_{yy}(x,y)|,$$

$$M_{21} = \max_{(x,y)\in K} |f_{xxy}(x,y)|, \quad M_{12} = \max_{(x,y)\in K} |f_{xyy}(x,y)|, \tag{7}$$

$$M_{30} = \max_{(x,y)\in K} |f_{xxx}(x,y)|, \quad M_{03} = \max_{(x,y)\in K} |f_{yyy}(x,y)|.$$

We let denote

$$y^{(m)} = P_m(f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, ..., f_{x^n}, f_{x^{n-1}y}, ..., f_{y^n}), \tag{8}$$

where $f_{x^\alpha y^\beta}$ denotes $\frac{\partial^{\alpha+\beta} f[x,y(x)]}{\partial x^\alpha y^\beta}$, P_m is a polinomial of degree m. For example,

$$P_1 = f, \; P_2 = f_x + f_y f, \; P_3 = f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x.$$

Further, we use the following notation as well:

$$l_m = P_m(M_0, M_{10}, M_{01}, M_{11}, M_{12}, M_{02}, ..., M_{n0}, M_{n-1,1}, ..., M_{0n}).$$

2 Main result

First, we evaluate the derivative of the approximate solution:

$$\dot{\widetilde{y}}(x) = f[\delta_s, \widetilde{y}(\delta_s)] + (s - \delta_s)(f_y f + f_x)[\delta_s, \widetilde{y}(\delta_s)] +$$

$$+ \frac{(s-\delta_s)^2}{2}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)f + f_y f_x\right)[\delta_s, \widetilde{y}(\delta_s)], \tag{9}$$

that is, the approximate solution $\widetilde{y}(x)$ consists of a cubic spline. Since $0 \le s - \delta_s \le h$, equation (9) leads to the following estimate:

$$\left|\dot{\widetilde{y}}(x)\right| \le l_1 + l_2 h + l_3 \frac{h^2}{2}.$$

Theorem. For the approximate solution, the estimate

$$|y_*(x) - \widetilde{y}(x)| \le (L_0 + L_1 h + L_2 h^2) h^3 \frac{e^{M_{01}(x-x_0)} - 1}{M_{01}}$$

holds true, where

$L_0 = \frac{1}{6}((3M_{21} + M_{01}M_{11} + M_{10}M_{02})l_1 + M_{02}l_1l_2 + (3M_{12} + M_{01}M_{02}){l_1}^2 +$

$+M_{03}{l_1}^3 + M_{30} + M_{10}M_{11}),$

$L_1 = \frac{1}{24}((M_{21} + M_{02}M_{10})l_2 + (2M_{12} + M_{02}M_{01})l_1l_2 + M_{11}l_3 + M_{02}l_1l_3 + {l_1}^2l_2M_{03}),$

$L_2 = \frac{1}{120}((M_{02}M_{10} + M_{21})l_3 + (2M_{12} + M_{02}M_{01})l_1l_3 + M_{03}{l_1}^2l_3).$

Proof. Using the equation

$$y_*(x) = y_0 + \int\limits_{x_0}^{x} f[s, y_*(s)]ds$$

for the exact solution, and equation (6) for the approximate solution yields

$$|y_*(x) - \widetilde{y}(x)| \leq \int\limits_{x_0}^{x} |\{f[s, y_*(s)] - f[s, \widetilde{y}(s)]\}|ds + \int\limits_{x_0}^{x} |I(s)|\, ds,$$

where

$$I(s) = f[s, \widetilde{y}(s)] - f[\delta_s, \widetilde{y}(\delta_s)] - (s - \delta_s)\,(f_y f + f_x)\,[\delta_s, \widetilde{y}(\delta_s)] -$$

$$- \frac{(s-\delta_s)^2}{2}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right)[\delta_s, \widetilde{y}(\delta_s). \qquad (10)$$

We transform the first two terms of the expression (10) as follows:

$$f[s, \widetilde{y}(s)] - f[\delta_s, \widetilde{y}(\delta_s)] = \int\limits_{\delta_s}^{s} \frac{d\,[f[r, \widetilde{y}(r)]]}{dr}dr = \int\limits_{\delta_s}^{s} \{f_x[r, \widetilde{y}(r)] + f_y[r, \widetilde{y}(r)]\widetilde{y}[r]\}\, dr =$$

$$= \int\limits_{\delta_s}^{s} f_x[r, \widetilde{y}(r)]dr + \int\limits_{\delta_s}^{s} f_y[r, \widetilde{y}(r)]f[\delta_r, \widetilde{y}(\delta_r)]dr +$$

$$+ \int\limits_{\delta_s}^{s} (r - \delta_r)\,(f_y f + f_x)\,[\delta_r, \widetilde{y}(\delta_r)]f_y[r, \widetilde{y}(r)]dr +$$

$$+ \int\limits_{\delta_s}^{s} \frac{(r-\delta_r)^2}{2}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2 f + f_y f_x\right)[\delta_r, \widetilde{y}(\delta_r)]f_y[r, \widetilde{y}(r)]dr.$$

We put the result into (10) and perform one of the following groupings:

$$\int\limits_{\delta_s}^{s} \{f_x[r, \widetilde{y}(r)] - f_x[\delta_s, \widetilde{y}(\delta_s)]\}dr = \int\limits_{\delta_s}^{s} \left(\int\limits_{\delta_s}^{r} \frac{d\,[f_x[u, \widetilde{y}(u)]]}{du}du \right) dr =$$

$$= \int\limits_{\delta_s}^{s} \int\limits_{\delta_s}^{r} \{f_{xx}[u, \widetilde{y}(u)] + f_{xy}[u, \widetilde{y}(u)]f[\delta_u, \widetilde{y}(\delta_u)]\}dudr +$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\{(u-\delta_u)\left(f_y f+f_x\right)[\delta_u,\widetilde{y}(\delta_u)]f_{xy}[u,\widetilde{y}(u)]\}dudr+ \quad (11)$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left\{\frac{(u-\delta_u)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_u,\widetilde{y}(\delta_u)]f_{xy}[u,\widetilde{y}(u)]\right\}dudr.$$

We can evaluate the second and third integrals in (11) in the following order (here $\sigma_r=\sigma_s$):

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}|(u-\delta_u)P_2f_{xy}[u,\widetilde{y}(u)]|dudr\le M_{11}l_2\frac{h^3}{6},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left|\frac{(u-\delta_u)^2}{2}P_3f_{xy}[u,\widetilde{y}(u)]\right|dudr\le M_{11}l_3\frac{h^4}{24}.$$

In a similar fashion we have

$$\int\limits_{\delta_s}^{s}\{f_y[r,\widetilde{y}(r)]-f_y[\delta_s,\widetilde{y}(\delta_s)]\}f[\delta_r,\widetilde{y}(\delta_r)]dr=\int\limits_{\delta_s}^{s}\left(\int\limits_{\delta_s}^{r}\frac{d\left[f_y[u,\widetilde{y}(u)]\right]}{du}du\right)f[\delta_r,\widetilde{y}(\delta_r)]dr=$$

$$=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\{f_{yy}[u,\widetilde{y}(u)]f[\delta_u,\widetilde{y}(\delta_u)][\delta_r,\widetilde{y}(\delta_r)]+f_{yx}[u,\widetilde{y}(u)]f[\delta_r,\widetilde{y}(\delta_r)]\}dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(u-\delta_u)\left(f_yf+f_x\right)[\delta_u,\widetilde{y}(\delta_u)]f_{yy}[u,\widetilde{y}(u)]f[\delta_r,\widetilde{y}(\delta_r)]dudr+ \quad (12)$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\frac{(u-\delta_u)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_u,\widetilde{y}(\delta_u)]f_{yy}[u,\widetilde{y}(u)]f[\delta_r,\widetilde{y}(\delta_r)]dudr.$$

We can evaluate the second and third integrals in the right hand side of (12) as follows:

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}|(u-\delta_u)P_2f_{yy}[\delta_u,\widetilde{y}(\delta_u)]f[\delta_r,\widetilde{y}(\delta_r)]|dudr\le M_{02}l_1l_2\frac{h^3}{6},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left|\frac{(u-\delta_u)^2}{2}P_3f_{yy}[u,\widetilde{y}(u)]f[\delta_r,\widetilde{y}(\delta_r)]\right|dudr\le M_{02}l_1l_3\frac{h^4}{24}.$$

After that, we group the rest terms in $I(s)$:

$$\int\limits_{\delta_s}^{s}(r-\delta_r)\left(f'_yf+f_x\right)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)]dr+$$

$$+\int\limits_{\delta_s}^{s}\frac{(r-\delta_r)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)]dr-$$

$$-\frac{(s-\delta_s)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_s,\widetilde{y}(\delta_s)]=$$

$$=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_yf+f_x)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)]dudr- \tag{13}$$

$$-\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_s,\widetilde{y}(\delta_s)]dudr+$$

$$+\int\limits_{\delta_s}^{s}\frac{(r-\delta_r)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)]dr.$$

We evaluate the last integral on the right hand side of (13) as follows:

$$\int\limits_{\delta_s}^{s}\left|\frac{(r-\delta_r)^2}{2}P_3f_y[r,\widetilde{y}(r)]\right|dr\leq M_{01}l_3\frac{h^3}{6}.$$

We group the unestimated terms of the expressions (11), (12) and the appropriate terms of the expression (13). We write the result in the following form:

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\{f_{xx}[u,\widetilde{y}(u)]+f_{xy}[u,\widetilde{y}(u)]f[\delta_u,\widetilde{y}(\delta_u)]\}dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\{f_{yy}[u,\widetilde{y}(u)]f[\delta_u,\widetilde{y}(\delta_u)][\delta_r,\widetilde{y}(\delta_r)]+f_{yx}[u,\widetilde{y}(u)]f[\delta_r,\widetilde{y}(\delta_r)]\}dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}((f_yf+f_x)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)])dudr-$$

$$-\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left(\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_s,\widetilde{y}(\delta_s)]\right)dudr+$$

$$+\int\limits_{\delta_s}^{s}\frac{(r-\delta_r)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_r,\widetilde{y}(\delta_r)]f_y[r,\widetilde{y}(r)]dr=$$

$$=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_{xx}[u,\widetilde{y}(u)]-f_{xx}[\delta_s,\widetilde{y}(\delta_s)])dudr+$$

$$+2\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_{xy}[u,\widetilde{y}(u)]-f_{xy}[\delta_s,\widetilde{y}(\delta_s)])f[\delta_r,\widetilde{y}(\delta_r)]dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_y[r,\widetilde{y}(r)]-f_y[\delta_s,\widetilde{y}(\delta_s)])(f_yf)[\delta_s,\widetilde{y}(\delta_s)]dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_y[r,\widetilde{y}(r)]-f_y[\delta_s,\widetilde{y}(\delta_s)])f_x[\delta_r,\widetilde{y}(\delta_r)]dudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_{yy}[u,\widetilde{y}(u)]-f_{yy}[\delta_s,\widetilde{y}(\delta_s)])f^2[\delta_s,\widetilde{y}(\delta_s)]dudr.$$

We denote the resulting integrals by a_1, a_2, a_3, a_4, a_5 and simplify each of them. We transform the integral a_1 as follows:

$$a_1=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_{xx}[u,\widetilde{y}(u)]-f_{xx}[\delta_s,\widetilde{y}(\delta_s)])dudr=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\left(\int\limits_{\delta_s}^{u}\frac{d[f_{xx}[u,\widetilde{y}(u)]]}{dv}dv\right)dudr=$$

$$=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left(f_{xxx}[v,\widetilde{y}(v)]+f_{xxy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)]\right)dvdudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}(v-\delta_v)\,(f_yf+f_x)[\delta_v,\widetilde{y}(\delta_v)]f_{xxy}[v,\widetilde{y}(v)]dvdudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\frac{(v-\delta_v)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_v,\widetilde{y}(\delta_v)]f_{xxy}[v,\widetilde{y}(v)]dvdudr.$$

Then, we will evaluate each of these integrals.

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|f_{xxx}[v,\widetilde{y}(v)]+f_{xxy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)]\right|dvdudr\leq(M_{30}+M_0M_{21})\frac{h^3}{6},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}|(v-\delta_v)\,P_2f_{xxy}[v,\widetilde{y}(v)]|dvdudr\leq M_{21}l_2\frac{h^4}{24},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|\frac{(v-\delta_v)^2}{2}P_3f_{xxy}[v,\widetilde{y}(v)]\right|dvdudr\leq M_{21}l_3\frac{h^5}{120}.$$

Similarly, we simplify the expression a_2.

$$a_2 = 2\int_{\delta_s}^{s}\int_{\delta_s}^{r}(f_{xy}[u,\widetilde{y}(u)] - f_{xy}[\delta_s,\widetilde{y}(\delta_s)])f[\delta_s,\widetilde{y}(\delta_s)]dudr =$$

$$= 2\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}(f_{xyx}[v,\widetilde{y}(v)] + f_{xyy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)])\,f[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+2\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}(v-\delta_v)\,(f_yf + f_x)[\delta_v,\widetilde{y}(\delta_v)]f_{xyy}[v,\widetilde{y}(v)]f[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+2\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\frac{(v-\delta_v)^2}{2}\left(f_{xx} + 2f_{xy}f + f_{yy}f^2 + (f_y)^2f + f_yf_x\right)[\delta_v,\widetilde{y}(\delta_v)]$$

$$f_{xyy}[v,\widetilde{y}(v)]f[\delta_s,\widetilde{y}(\delta_s)]dvdudr.$$

Next, we evaluate each of these integrals.

$$2\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}|(f_{xyx}[v,\widetilde{y}(v)] + f_{xyy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)])\,f[\delta_s,\widetilde{y}(\delta_s)]|\,dvdudr \le$$

$$\le 2(M_{21}M_0 + {M_0}^2M_{12})\frac{h^3}{6},$$

$$2\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}|(v-\delta_v)\,P_2f_{xyy}[v,\widetilde{y}(v)]f[\delta_s,\widetilde{y}(\delta_s)]|dvdudr \le 2M_{12}l_1l_2\frac{h^4}{24},$$

$$\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\left|(v-\delta_v)^2P_3f_{xyy}[v,\widetilde{y}(v)]f[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr \le 2M_{12}l_1l_3\frac{h^5}{120}.$$

We now simplify the expression a_3.

$$a_3 = \int_{\delta_s}^{s}\int_{\delta_s}^{r}(f_y[r,\widetilde{y}(r)] - f_y[\delta_s,\widetilde{y}(\delta_s)])(f_yf)[\delta_s,\widetilde{y}(\delta_s)]dudr =$$

$$= \int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}(f_{yx}[v,\widetilde{y}(v)] + f_{yy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)])\,(f_yf)[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}(v-\delta_v)\,(f_yf + f_x)[\delta_v,\widetilde{y}(\delta_v)]f_{yy}[v,\widetilde{y}(v)](f_yf)[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\frac{(v-\delta_v)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_v,\widetilde{y}(\delta_v)]$$

$$f_{yy}[v,\widetilde{y}(v)](f_yf)[\delta_s,\widetilde{y}(\delta_s)]dvdudr.$$

Then, we evaluate each of these integrals.

$$\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\left|\left(f_{yx}[v,\widetilde{y}(v)]+f_{yy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)]\right)(f_yf)[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le$$

$$\le(M_0M_{01}M_{11}+{M_0}^2M_{01}M_{02})\frac{h^3}{6},$$

$$\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}|(v-\delta_v)\,P_2f_{yy}[v,\widetilde{y}(v)](f_yf)[\delta_s,\widetilde{y}(\delta_s)]|dvdudr\le M_{01}M_{02}l_1l_2\frac{h^4}{24},$$

$$\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\left|\frac{(v-\delta_v)^2}{2}P_3f_{yy}[v,\widetilde{y}(v)](f_yf)[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le M_{01}M_{02}l_1l_3\frac{h^5}{120}.$$

We now simplify the expression.

$$a_4=\int_{\delta_s}^{s}\int_{\delta_s}^{r}(f_y[r,\widetilde{y}(r)]-f_y[\delta_s,\widetilde{y}(\delta_s)])f_x[\delta_r,\widetilde{y}(\delta_r)]dudr=$$

$$=\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\left(f_{yx}[v,\widetilde{y}(v)]+f_{yy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)]\right)f_x[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}(v-\delta_v)\,(f_yf+f_x)[\delta_v,\widetilde{y}(\delta_v)]f_{yy}[v,\widetilde{y}(v)]f_x[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\frac{(v-\delta_v)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+{f'}_yf_x\right)[\delta_v,\widetilde{y}(\delta_v)]$$

$$f_{yy}[v,\widetilde{y}(v)]f_x[\delta_s,\widetilde{y}(\delta_s)]dvdudr.$$

Let us evaluate each of these integrals.

$$\int_{\delta_s}^{s}\int_{\delta_s}^{r}\int_{\delta_s}^{u}\left|\left(f_{yx}[v,\widetilde{y}(v)]+f_{yy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)]\right)f_x[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le$$

$$\le(M_{10}M_{11}+M_0M_{10}M_{02})\frac{h^3}{6},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}|(v-\delta_v)\,P_2f_{yy}[v,\widetilde{y}(v)]f_x[\delta_s,\widetilde{y}(\delta_s)]|dvdudr \le M_{10}M_{02}l_2\frac{h^4}{24},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|\frac{(v-\delta_v)^2}{2}P_3f_{yy}[v,\widetilde{y}(v)]f_x[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr \le M_{10}M_{02}l_3\frac{h^5}{120}.$$

Finally, we simplify the expression:

$$a_5=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}(f_{yy}[u,\widetilde{y}(u)]-f_{yy}[\delta_s,\widetilde{y}(\delta_s)])f^2[\delta_s,\widetilde{y}(\delta_s)]dudr=$$

$$=\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}(f_{yyx}[v,\widetilde{y}(v)]+f_{yyy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)])\,f^2[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}(v-\delta_v)\,(f_yf+f_x[\delta_v,\widetilde{y}(\delta_v)]f_{yyy}[v,\widetilde{y}(v)]f^2[\delta_s,\widetilde{y}(\delta_s)]dvdudr+$$

$$+\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\frac{(v-\delta_v)^2}{2}\left(f_{xx}+2f_{xy}f+f_{yy}f^2+(f_y)^2f+f_yf_x\right)[\delta_v,\widetilde{y}(\delta_v)]$$

$$f_{yyy}[v,\widetilde{y}(v)]f^2[\delta_s,\widetilde{y}(\delta_s)]dvdudr.$$

Let us evaluate each of these integrals:

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|(f_{yyx}[v,\widetilde{y}(v)]+f_{yyy}[v,\widetilde{y}(v)]f[\delta_v,\widetilde{y}(\delta_v)])\,f^2[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le$$

$$\le({M_0}^2M_{12}+{M_0}^3M_{03})\frac{h^3}{6},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|(v-\delta_v)\,P_2f_{yyy}[v,\widetilde{y}(v)]f^2[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le M_{03}{l_1}^2l_2\frac{h^4}{24},$$

$$\int\limits_{\delta_s}^{s}\int\limits_{\delta_s}^{r}\int\limits_{\delta_s}^{u}\left|\frac{(v-\delta_v)^2}{2}P_3f_{yyy}[v,\widetilde{y}(v)]f^2[\delta_s,\widetilde{y}(\delta_s)]\right|dvdudr\le M_{03}{l_1}^2l_3\frac{h^5}{120}.$$

We have estimated all the above expressions. These estimates allow us to write the following inequality $\int\limits_{x_0}^{x}|I(s)|\,ds\le Ch^3(x-x_0)$, where $C=L_0+L_1h+L_2h^2$.

Hence,

$$|y_*(x)-\widetilde{y}(x)|\le Ch^3(x-x_0)+\int\limits_{x_0}^{x}|f[s,y_*(s)]-f[s,\widetilde{y}(s)]|ds.$$

According to the Lipschitz condition (it is not difficult to see that $L = M_{01}$ [6])

$$|f[s, y_*(s)] - f[s, \widetilde{y}(s)]| \leq M_{01} \left| y_*(s) - \widetilde{y}(s) \right| .$$

Therefore, $|y_*(x) - \widetilde{y}(x)| \leq Ch^3(x - x_0) + M_{01} \int\limits_{x_0}^{x} |y_*(s) - \widetilde{y}(s)| ds$. Applying Gronoul's lemma to this inequality [9], we get

$$|y_*(x) - \widetilde{y}(x)| \leq Ch^3 \frac{e^{M_{01}(x-x_0)} - 1}{M_{01}}.$$

□

3 Conclusion

Estimating the accuracy of the approximate solution of the Cauchy problem for an ordinary differential equation is important for both theoretical research and practical applications. In this case, it is not enough to find the values of the solution in the argument grid, but it is of particular importance to estimate the difference between the approximate and exact solutions in the entire range under consideration. The result obtained in the article solves this problem.

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