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LIMIT THEOREMS FOR WEAKLY DEPENDENT RANDOM VARIABLES WITH VALUES IN STABLE TYPE p BANACH SPACES

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Abstract

We consider stable type p Banach spaces. We extend results known for independent random variables to the mixing random variables.

In particular we prove moment in equalities, low of large numbers and almost sure convergence of the series in the case of mixing random variables.

Keywords: *Rademacher type p Banach space, a stable type p Banach space, mixing condition, law of large numbers, random series.*

Mathematics Subject Classification (2020): *60B11, 60B12.*

Introduction

Limit theorems for random variables with values in Banach spaces were studied by many authors (see for instance [2, 3, 5], [11]-[17], [19]-[25] and references therein). In particular such limit theorems are important in functional data analysis and in mathematical statistics. It is known that many statistical estimators can be expressed as a function of sums of Banach space-valued random variables and limit theorems in Banach spaces allow us to investigate asymptotic properties of statistical estimators.

Validity of limit theorems in Banach space depends on geometrical structure of the space. As consequence of this fact there are several types of Banach spaces, see for instance [23]. In this paper we will consider two types of the Banach spaces. Namely, we will consider Rademacher type p and a stable type p Banach spaces.

Let B be a separable Banach spaces with a norm $\|\cdot\|$ and $\{X_n,\, n\geq 1\}$ be a sequence of independent random variables with values in B. Now we will give a definition of Rademacher type p Banach space.

Definition 1. *We say that B is Rademacher type $p\,(1\leq p\leq 2)$ Banach space if for any finite collection of $B-$valued independent random variables $X_1,\, X_2,\, ...,\, X_n$ with $EX_i=0,\; E\left\|X_i\right\|^p<\infty$ there exists a constant $C=C(p,\,B)>0$ such that*

$$E\left\|\sum_{i=1}^{n}X_i\right\|^p\leq C\sum_{i=1}^{n}E\left\|X_i\right\|^p.$$

Obviously any separable Banach space is Rademacher type 1 space. L_p, l_p spaces are Rademacher type 2 Banach spaces for $p\geq 2$ and L_p, l_p are Rademacher type p spaces if $1\leq p\leq 2$. c_0 space is not Rademacher type p space for any $p\in(1,\;2]$.

By θ we denote a real valued stable random variable with characteristic function

$$E\exp(it\theta) = \exp\left(-\left|t\right|^p\right)$$

$\{\theta_i,\ i \geq 1\}$ is a sequence of independent copies of θ.

Now we give a definition of stable type p Banach spaces.

Definition 2. *We say that B is a stable type $p\,(0 < p \leq 2)$ Banach space if for each $q < p$ there exists a constant $C > 0$ such that for all integers n and any $x_1,\ x_2,\ ...,\ x_n \in B$ the following inequality holds*

$$\left(E\left\|\sum_{i=1}^{n}\theta_i x_i\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}. \tag{1}$$

There are several equivalent definitions of Rademacher type p and stable type p Banach spaces (see [15], [23]). The following facts are known (see [15, 23, 17, 19]):

1) Separable Banach space B is of stable type p for all $p \in (0,\ 1)$.

2) For $1 \leq p \leq 2$, B is of stable type p if and only if B is of Rademacher type p_1 for some $p_1 \in (p,\ 2]$.

3) B is of stable type 2 if and only if B is of Rademacher type 2.

   Consequently:

4) If B is of stable type p for some $p \in [1,\ 2]$, then B is of Rademacher type p.

5) If B is of stable type p for some $p \in [1,\ 2]$, then B is of stable q for all $q \in (0,\ p)$.

6) If B is of stable type p for some $p \in [1,\ 2)$, then B is of stable q for some $q \in (p,\ 2]$.

It is known that B is of stable type p if and only if for all sequences $x_1,\ x_2,\ ...$ of elements of B such that $\sum\limits_{i=1}^{\infty}\|x_i\|^p < \infty$ the series

$$\sum_{i=1}^{\infty}\theta_i x_i$$

converges almost surely, see [23].

If B is of stable type p, then for independent random variables $\xi_1,\ \xi_2,\ ...,\ \xi_n$ with characteristic functions $\exp\left(-\alpha_i\left|t\right|^p\right)$, $i = 1,\ 2,\ ...,\ n$ and any $x_1,\ x_2,\ ...,\ x_n \in B$ the following inequality holds (see [23])

$$\left(E\left\|\sum_{i=1}^{n}\xi_i x_i\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{n}\alpha_i\|x_i\|^p\right)^{1/p} \tag{2}$$

where $q < p$ and $C = C(q) > 0$.

By $\{r_i,\, i \geq 1\}$ we denote Rademacher sequence, i.e. $\{r_i,\, i \geq 1\}$ is a sequence of iid random variables with $P\left(r_i = \pm 1\right) = \frac{1}{2}$.

Now we will formulate some known results for the case of independent random variables.

Theorem 1 ([23, 24]). *The following properties of a Banach space B are equivalent,*

(i) The space B is of Rademacher-type p;

(ii) If (X_i) is a sequence of independent, zero-mean random vectors in B, then the condition

$$\sum_{i=1}^{\infty} E\phi_p\left(\|X_i\|\right) < \infty,\ \ \phi_p(t) := \min\left\{t^p,\ t\right\},\ \ t \geq 0,$$

implies the almost sure convergence of the random series $\sum\limits_i X_i$;

(iii) If (X_i) is a sequence of independent, zero-mean random vectors in B, then the convergence of the series $\sum\limits_i E\left\|X_i\right\|^p$ implies the almost sure convergence of the random series $\sum\limits_i X_i$;

(iv) For any $(x_n) \subset B$ with $\sum\limits_i \|x_i\|^p < \infty$, the random series $\sum\limits_i r_i x_i$ converges almost surely, and in L_p.

Next theorem is related to the almost sure convergence of some special random series in spaces of stable-type p.

Theorem 2 ([23, 17]). *Let $1 \leq p < 2$. The following properties of a Banach space B are equivalent:*

(i) The space B is of stable-type p;

(ii) For any sequence $(x_i) \subset B$ with $\sum\limits_i \|x_i\|^p < \infty$, and any sequence of independent and identically distributed $p-$stable random variables (θ_i), the series $\sum\limits_i \theta_i x_i$ converges almost surely, and in L_q, if $q < p$;

(iii) For any bounded sequence $(x_n) \subset B$, the series $\sum\limits_n n^{-1/p} r_n x_n$ converges almost surely;

(iv) For any bounded sequence $(x_n) \subset B$, there exists a choice of $\varepsilon_n = \pm 1$, such that $\sum\limits_n n^{-1/p} \varepsilon_n x_n$ converges.

Theorem 3 ([23, 24]). *Let $p \in (1,\, 2]$. The following properties of a Banach space B are equivalent:*

(i) The space B is of Rademacher-type p;

(ii) *For each sequence (X_n) of independent, zero-mean random vectors in B, the convergence of the series $\sum_n n^{-p} E\|X_n\|^p$ implies that*

$$\lim_{n\to\infty} \frac{X_1 + X_2 + ... + X_n}{n} = 0, \quad a.s.;$$

(iii) *There exists a constant $C > 0$ such that, for each finite sequence $(x_i) \subset B$,*

$$\frac{1}{n} E \left\| \sum_{i=1}^{n} r_i x_i \right\| \le C \left(\sum_{i=1}^{n} \frac{\|x_i\|^p}{i^p} \right)^{1/p};$$

(iv) *There exists a constant $C > 0$ such that, for each finite sequence (X_n) of independent, zero-mean random vectors in B,*

$$\frac{1}{n} E \left\| \sum_{i=1}^{n} X_i \right\| \le C \left(\sum_{i=1}^{n} \frac{E\|X_i\|^p}{i^p} \right)^{1/p}.$$

Theorem 4 ([23, 16])**.** *Let $1 \le p < 2$. The following properties of a Banach space B are equivalent:*

(i) *The space B is of stable-type p;*

(ii) *For each sequence (X_n) of symmetric, independent, and identically distributed random vectors in B,*

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} X_n = 0,$$

in probability, if, and only if,

$$\lim_{n\to\infty} nP\left(\|X_1\| > n^{1/p}\right) = 0.$$

Theorem 5 ([23])**.** *Let $1 \le p \le 2$. The following properties of a Banach space B are equivalent:*

(i) *The space B is of stable-type p;*

(ii) *For any sequence $(x_i) \subset B$, such that $\sum_i i^{-p}\|x_i\|^p < \infty$, and each sequence (θ_i) of independent and identically distributed $p-$stable random variables,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \theta_i x_i = 0, \ \ a.s., \text{ and also in } L_q, \ \ q < p;$$

(iii) *There exists a constant C such that, for any $(x_i) \subset B$, and (θ_i), as above,*

$$\frac{1}{n} \left(E \left\| \sum_{i=1}^{n} \theta_i x_i \right\|^{p/2} \right)^{2/p} \le C \left(\sum_{i=1}^{n} i^{-p}\|x_i\|^p \right)^{1/p}.$$

Theorem 6 ([23]). *Let $1 \le p < 2$. The following properties of a Banach space B are equivalent:*

(i) The space B is of stable-type p;

(ii) For each bounded sequence $(x_n) \subset B$,

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} r_k x_k = 0, \quad a.s.;$$

(iii) For each bounded sequence $(x_n) \subset B$ there exists a choice of $\varepsilon_i = \pm 1$ such that

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} \varepsilon_k x_k = 0.$$

Only few of above results were extended to the case of mixing random variables.

For the sequence $\{\theta_i,\, i \ge 1\}$ mixing coefficients are defined as following

$$\psi(k) = \sup\left\{ \frac{|P(AB) - P(A)P(B)|}{P(A)P(B)} \;:\; A \in F_1^n,\, B \in F_{n+k}^{\infty},\, n \in N,\, P(A)P(B) > 0 \right\}$$

where F_a^b is a $\sigma-$ field generated by random variables θ_a, ..., θ_b.

We say that the sequence $\{\theta_i,\, i \ge 1\}$ is $\psi-$mixing, if $\lim\limits_{n\to\infty} \psi(n) = 0$.

Denote $S_n = \sum\limits_{i=1}^{n} X_i$.

In [21] the following theorem was proved for weakly dependent random variables.

Theorem 7. *Let $\{X_n,\ n \ge 1\}$ be a sequence of random variables with values in B, for which the following conditions hold*

$$EX_k = 0, \quad E\left\|X_k\right\|^p < \infty, \quad k = 1, 2, ...,\ 1 \le p \le 2$$

$$\psi = \prod_{k=1}^{\infty} (1 + \psi(k)) < \infty,\ \psi(1) < 1.$$

Then following statements are equivalent:

1) B is of Rademacher type p space;

2) there is a constant $C(B,\, p,\, \psi)$ such that for any sequence $\{X_n,\ n \ge 1\}$ satisfying the conditions of the theorem, we have

$$E\left\|S_n\right\|^p \le C(B, p, \psi) \sum_{k=1}^{n} E\left\|X_k\right\|^p; \tag{3}$$

3) for all sequences $\{X_n,\ n \geq 1\}$ satisfying the conditions of the theorem and

$$\sum_{k=1}^{\infty} k^{-p} E \left\| X_k \right\|^p < \infty,$$

the strong law of large numbers holds, that is, as $n \to \infty$

$$\frac{1}{n} S_n \to 0 \quad a.s.$$

1 Main results

In this section we will give our main results. Our main goal is to extend some of the results for independent random variables given above to the case of $\psi-$mixing random variables. Such theorems allow us to investigate sums of Banach space-valued stable random variables. Sums of weakly dependent real valued stable random variables were studied in [1], [6]-[10], [18].

Our main results are following theorems.

Theorem 8. *Let $\{\theta_i,\ i \geq 1\}$ be a $\psi-$mixing sequence of random variables with characteristic functions*

$$f_{\theta_i}(t) = \exp\left(-\alpha_i \left|t\right|^p\right), \quad \alpha_i > 0$$

and assume that

$$\psi = \prod_{k=1}^{\infty} \left(1 + \psi(k)\right) < \infty.$$

Then for any elements $x_1,\ x_2,\ ...,\ x_n$ of stable type p Banach space B there exists a constant $C = C(B,\ \psi,\ q)$ such that

$$\left(E \left\| \sum_{i=1}^{n} \theta_i x_i \right\|^q \right)^{1/q} \leq C \left(\sum_{i=1}^{n} \alpha_i^p \left\| x_i \right\|^p \right)^{1/p}.$$

Theorem 9. *The following statements are equivalent:*

1) B is a stable type p Banach space.

2) For any sequence $\{x_i,\ i \geq 1\}$ of elements of B such that

$$\sum_{i=1}^{\infty} \left\| x_i \right\|^p < \infty \tag{4}$$

and any sequence $\{\theta_i,\ i \geq 1\}$ of $\psi-$mixing identically distributed $p-$stable random variables with characteristic function $f_{\theta_i}(t) = \exp\left(-\left|t\right|^p\right)$ and

$$\sum_{k=1}^{\infty} \psi(k) < \infty,\ \ \psi(1) < 1$$

the series

$$\sum_{i=1}^{\infty} \theta_i x_i$$

converges almost surely and in L_q $q < p$.

3) *For any bounded sequence* $\{x_n,\ n \geq 1\}$ *of elements of* B *and any sequence* $\{r_n,\ n \geq 1\}$ *of* $\psi-$*mixing random variables with*

$$P\left(r_n = \pm\right) = \frac{1}{2},\ n = 1,\, 2,\, ...$$

$$\sum_{k=1}^{\infty} \psi(k) < \infty$$

the series

$$\sum_{n=1}^{\infty} \frac{r_n}{n^{1/p}} x_n$$

converges almost surely.

4) *For any bounded sequence* $\{x_n,\ n \geq 1\}$ *of elements of* B *there exists a choice of* $\varepsilon_n = \pm 1$ *such that the series*

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{n^{1/p}} x_n$$

converges.

Theorem 10. *Let* B *be a stable type* p $(1 \leq p \leq 2)$ *Banach space and* $\{X_i,\ i \geq 1\}$ *be a sequence of symmetric, identically distributed* $\psi-$*mixing random variables with values in* B*. Assume that the following conditions hold*

$$\sum_{k=1}^{\infty} \psi(k) < \infty,$$

$$\lim_{n \to \infty} nP\left(\|X_1\| > n^{1/p}\right) = 0. \tag{5}$$

Then as $n \to \infty$

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i \to 0 \quad \textit{in probability.}$$

In the following proposition we slightly improve the statement 2) of Theorem 9.

Proposition 1. *Let* $\theta_1,\, \theta_2,\, ...,\, \theta_n$ *be a finite sequence of stable random variables with characteristic functions*

$$f_{\theta_i}(t) = \exp\left(-\alpha_i \left|t\right|^p\right),\ \alpha_i > 0$$

and

$$\psi = \prod_{k=1}^{n} (1 + \psi(k)) < \infty, \ \psi(1) < 1$$

and B is of stable type p Banach space.

In addition if

$$\sum_{i=1}^{n} \alpha_i^p \|x_i\|^p < \infty.$$

Then the series

$$\sum_{i=1}^{\infty} \theta_i x_i \tag{6}$$

converges almost surely.

2 Proofs of theorems

Proof of Theorem 8. We will use the following lemma.

Lemma 1 ([25]). *For any sequence $\{X_n, n \geq 1\}$ random variables with $\psi-$mixing and with values in B there is a sequence $\{Y_n, n \geq 1\}$ of independent random variables with values in B such that X_k and Y_k are identically distributed for $k = 1, 2, ...$ and the following inequality holds*

$$|P(S_n \in A) - P(Z_n \in A)| \leq \prod_{k=1}^{n} (1 + \psi(k)) P(Z_n \in A),$$

where $Z_n = Y_1 + Y_2 + \ ... \ + Y_n$ and A is any Borel set in B.

Denote by $S_n = \sum\limits_{i=1}^{n} \theta_i x_i$ and $Z_n = \sum\limits_{i=1}^{n} \theta'_i x_i$ where $\{\theta'_i, i \geq 1\}$ is a sequence of random variables satisfying conditions of Lemma 1.

From Lemma 1 it follows that

$$|P(\|S_n\| > x) - P(\|Z_n\| > x)| \leq \prod_{k=1}^{n} (1 + \psi(k)) P(\|Z_n\| > x),$$

where $Z_n = Y_1 + Y_2 + \ ... \ + Y_n$, $Y_k = \theta'_k x_k$.

Using the last inequality, we get

$$|E\|S_n\|^q - E\|Z_n\|^q| = \int_0^\infty pt^{p-1} (P(\|S_n\| > t) - P(\|Z_n\| > t)) \, dt \leq$$

$$\leq \int_0^\infty pt^{p-1} (P(\|S_n\| > t) - P(\|Z_n\| > t)) \, dt$$

$$\leq \int_0^\infty pt^{p-1} \psi P(\|Z_n\| > t) \, dt = \psi E\|Z_n\|^q,$$

from here and from (1), taking into account that X_k and Y_k are identically distributed for $k = 1, 2, ...$ we have

$$E\left\|S_n\right\|^q \leq (1+\psi)C(B,\, q)E\left\|Z_n\right\|^q \leq C(B,\, q,\, \psi)\left(\sum_{k=1}^{n}\alpha_k^p\left\|x_k\right\|^p\right)^{q/p}.$$

where $\psi = \prod_{k=1}^{n}(1+\psi(k))$.

The latter implies the statement of the theorem. Theorem is proved. □

Proof of Theorem 9. We first prove 1) ⇒ 2).

From the inequality

$$\left(E\left\|\sum_{i=1}^{n}\theta_i x_i\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{n}\left\|x_i\right\|^p\right)^{1/p}$$

where $C = C(q,\, \psi,\, B)$, $0 < q < p$ it follows that

$$E\left\|\sum_{i=1}^{n}\theta_i x_i\right\|^q < \infty.$$

The latter and (4) imply the convergence of

$$\sum_{i=1}^{\infty}\theta_i x_i \text{ in } L_q,\ \ q < p \tag{7}$$

(7) implies convergence of

$$\sum_{i=1}^{\infty}\theta_i x_i$$

in probability.

It remains to use the following lemma.

Lemma 2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables with values in the Banach space B. Assume that $\psi(1) < 1$. Then following statements are equivalent:*

1) $\sum_{k=1}^{\infty} X_k$ *converges in probability;*

2) $\sum_{k=1}^{\infty} X_k$ *converges with probability 1.*

Lemma 2 was proven in [4] (Theorem 2.1.12, pp. 13-14) for one-dimensional random variables, but the proof also holds in the case of random variables with values in Banach spaces, see also [21].

Now we prove 1) ⇒ 3). We will use the following lemma.

Lemma 3. *Let B be a separable Rademacher type p Banach space. Suppose that $\{X_n,\ n \geq 1\}$ is a $\psi-$mixing sequence of $B-$valued random variables with*

$$EX_i = 0, \quad E\left\|X_i\right\|^p < \infty,$$

$$\sum_{k=1}^{\infty} \psi(k) < \infty,$$

$$\sum_{i=1}^{\infty} E\left\|X_i\right\|^p < \infty.$$

Then the series

$$\sum_{i=1}^{\infty} X_i$$

converges almost surely.

Proof. The proof of this lemma almost the same as the proof of implication $(i) \Rightarrow (ii)$ in the Theorem 1. It needs to use Theorem 7, more exactly inequality (3). □

A space B is a stable type p and Rademacher type q for some $q > p$. Since

$$\sum_{n=1}^{\infty} \frac{\|x_n\|^q}{n^{q/p}} < \infty$$

from Lemma 3 follows that the series

$$\sum_{n=1}^{\infty} \frac{r_n}{n^{1/p}} x_n$$

converges almost surely.
An implication 3) ⇒ 4) is obvious. An implication 4) ⇒ 1) is proved in Theorem 2. The theorem is proved. □
Proof of Theorem 10. We will use the fact that if B is stable type p, then it is Rademacher type q, for some $p < q < 2$. We will use a method of proof of Theorem 1 [23] and give full proof for completeness.
Set

$$\bar{X}_k = X_k I\left(\|X_n\| \leq n^{1/p}\right),$$

$$Y_n = \sum_{k=1}^{n} \bar{X}_k, \ D_n = \left\{\sum_{i=1}^{n} X_i = Y_n\right\}$$

where $I(\cdot)$ is an indicator function.
We have

$$P\left(\frac{1}{n^{1/p}}\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon\right) = P\left(\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon n^{1/p}\right) \leq$$

$$\leq P\left(\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon n^{1/p},\, D_n\right) + P\left(\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon n^{1/p},\, \bar{D}_n\right) \leq$$

$$\leq P\left(D_n\right) P\left(\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon n^{1/p} \Big/ D_n\right) + P\left(\bar{D}_n\right) P\left(\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon n^{1/p} \Big/ \bar{D}_n\right) \leq$$

$$\leq P\left(\|Y_n\| \geq \varepsilon n^{1/p}\right) + P\left(\bar{D}_n\right) \leq$$

$$\leq \frac{1}{\varepsilon^q} E\left\|\frac{1}{n^{1/p}} Y_n\right\|^q + \sum_{i=1}^{n} P\left(\|X_i\| \geq n^{1/p}\right).$$

Now using Theorem 7 we get

$$P\left(\frac{1}{n^{1/p}}\left\|\sum_{i=1}^{n} X_i\right\| \geq \varepsilon\right) \leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{q/p}} \sum_{i=1}^{n} E\left\|X_i\right\|^q I\left(\|X_i\| \leq n^{1/p}\right) +$$

$$+\sum_{i=1}^{n} P\left(\|X_i\| \geq n^{1/p}\right) \leq \frac{C(\psi,\, q,\, B)n}{\varepsilon^q n^{q/p}} E\left\|X_1\right\|^q I\left(\|X_1\| \leq n^{1/p}\right) +$$

$$+nP\left(\|X_1\| > n^{1/p}\right).$$

The second summand above tends to 0 by the condition (5). It remains to prove that the first summand above tends to 0.

We have

$$\frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} E\left\|X_1\right\|^q I\left(\|X_1\| \leq n^{1/p}\right) \leq$$

$$\leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{i=1}^{n} \int_{i-1 \leq \|X_1\|^p \leq i} \|X_1\|^q \, dP \leq$$

$$\leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{i=1}^{n} i^{q/p} P\left(i-1 \leq \|X_1\|^p \leq i\right) \leq$$

$$\leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{i=1}^{n} \left(\sum_{j=1}^{i} j^{\frac{q}{p}-1}\right) P\left(i-1 \leq \|X_1\|^p \leq i\right) \leq$$

$$\leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{j=1}^{n} j^{\frac{q}{p}-1} P\left(\|X_1\|^p > j-1\right).$$

For any $\delta > 0$ condition (5) implies there existence of $j_0 \in N$ such that for $j > j_0$

$$jP\left(\|X_i\|^p \geq j-1\right) < \delta.$$

The latter implies

$$\frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{j=1}^{n} j^{\frac{q}{p}-1} P\left(\|X_1\|^p > j-1\right) \leq$$

$$\leq \frac{C(\psi,\, q,\, B)}{\varepsilon^q n^{\frac{q}{p}-1}} \sum_{j=1}^{j_0} j^{\frac{q}{p}-1} P\left(\|X_1\|^p > j-1\right) + \varepsilon^{-q}\delta n^{1-\frac{q}{p}} \sum_{j=j_0+1}^{n} j^{\frac{q}{p}-2}.$$

The second summand above tends to 0 because δ is arbitrary. Thus, we have

$$P\left(\frac{1}{n^{1/p}}\left\|\sum_{i=1}^{n} X_i\right\| > \varepsilon\right) \to 0 \text{ as } n \to \infty \text{ for any } \varepsilon > 0$$

and this completes the proof of theorem. □

Proof of Proposition 1. As in the proof of Theorem 8 using (2) we have

$$\left(E\left\|\sum_{i=1}^{n} \theta_i x_i\right\|^q\right)^{1/q} \leq C(\psi,\, q,\, B)\left(\sum_{i=1}^{n} \alpha_i^p \|x_i\|^p\right)^{1/p}.$$

Now almost sure convergence of the series (6) follows by the same arguments as in the implication 1) ⇒ 2) of Theorem 9. □

References

[1] Denker M. and Jakubowski A. Stable limit distributions for strongly mixing sequences. Statist. Probab. Lett. 8, pp. 477–483 (1989).

[2] Hoffmann-Jørgensen J. Sums of independent Banach space valued random variables, Studia Math. 52, pp. 159–186 (1974).

[3] Hoffmann-Jørgensen J., Pisier G. The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. 4, pp. 587–599 (1976).

[4] Iosifescu M., Theodorescu R. Random Processes and Learning. Springer Verlag. Berlin, (1969).

[5] Jajte R. On stable distributions in Hilbert space. Studia Math. XXX, pp. 63-71 (1968).

[6] Jakubowski A. and Kobus M. $p-$stable limit theorems for sums of dependent random vectors. J. Multivariate Anal. 29, pp. 219–251 (1989).

[7] Jakubowski A. On multidimensional domains of attraction for stationary sequences. Statist. Probab. Lett. 19, pp. 321–326 (1994).

[8] Jakubowski A. Minimal conditions in $p-$stable limit theorems. Stoch. Proc. Appl. 44, pp. 291–327 (1993).

[9] Jakubowski A. Minimal conditions in $p-$stable limit theorems - II. Stoch. Proc. Appl. 68, pp. 1–20 (1997).

[10] Johnson O. and Samworth R. Central limit theorem and convergence to stable laws in Mallows distance. Bernoulli 11(5), pp. 829–845 (2005).

[11] Jurek Z. and Urbanik K. Remarks on stable measures on Banach spaces. Colloquium mathematicum, Vol. 38, Fasc. 2, (1978).

[12] Korzeniowski A. On Marcinkiewicz SLLN in Banach spaces. Ann. Probab. 12, pp. 279–280 (1984).

[13] Klosowska M. The domain of attraction of a non-gaussian stable distribution in a Hilbert space. Colloq. Math. 32, pp. 127–136 (1972).

[14] Kuelbs J. and Mandrekar V. Domains of attraction of stable measures on a Hilbert space. Studia Math. 59, pp. 149–162 (1974).

[15] Ledoux M., Talagrand M. Probability in Banach Spaces: Isoperimetry and Processes. Springer, Berlin (1991).

[16] Marcus M.B. and Woyczynski W.A. Stable measures and central limit theorems in spaces of stable type. Transactions of the American Mathematical Society, Volume 251, pp. 71–102 (July 1979).

[17] Maurey B., Pisier G. Series de variables aleatoires vectorielles independantes et proprietes geometriques des espaces de Banach. Stud. Math. 58, pp. 45–90 (1976).

[18] Nadarajah S., Chan S. The exact distribution of the sum of stable random variables. Journal of Computational and Applied Mathematics. 349, pp. 187–196 (2019).

[19] Pisier G. Probabilistic methods in the geometry of Banach spaces. In: Probability and Analysis, Lectures Given at the 1st 1985 Session of the Centro Internazionale Matematico Estivo (C.I.M.E). Lecture Notes in Mathematics, Springer, Berlin, vol. 1206, pp. 167–241 (1986).

[20] Rosinski J. Remarks on Banach spaces of stable type. Probab. Math. Statist. 1, pp. 67–71 (1980).

[21] Sharipov O.Sh. On probabilistic characterization of Banach spaces by weakly dependent random elements. Uzb. Math. J. № 2, pp. 86–91 (1998). (in Russian)

[22] Woyczynski W.A. Geometry and martingales in Banach spaces, Part II: Independent increments. In: Kuelbs, J., Ney, P. (eds.) Probability on Banach Spaces. Advances in Probability and Related Topics, Dekker, New York, vol. 4, pp. 267–517 (1978).

[23] Woyczynski W.A. Geometry and martingales in Banach spaces. CRC Press; 1st edition, (2020).

[24] Woyczynski W.A. Random series and laws of large numbers in some Banach spaces. Teorya Veroyatnostej i Prim. 18, pp. 371–377 (1973).

[25] Zuparov T.M. Investigations on limit theorems for sums of weakly dependent Banach valued random variables. Doctor of Sciences thesis. Tashkent, 284 pp (1991). (in Russian)