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LOCAL AND 2-LOCAL DERIVATIONS ON SMALL DIMENSIONAL ZINBIEL ALGEBRAS

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Abstract

In the present paper we investigate local and 2-local derivations on small dimensional Zinbiel algebras. We give a description of derivations and local derivations on all three and four-dimensional Zinbiel algebras. Moreover, similar problem concerning 2-local derivations on all three and four-dimensional Zinbiel algebras are investigated.

Keywords: *Zinbiel algebras, derivation, local derivation, 2-local derivation.*
Mathematics Subject Classification (2010): *17A32, 17B30, 17B10.*

Introduction

In recent years non-associative analogues of classical constructions become of interest in connection with their applications in many branches of mathematics and physics. The notions of local and 2-local derivations are also become popular for some non-associative algebras such as Lie and Leibniz algebras.

The notions of local derivations were introduced in 1990 by R.V.Kadison [25] and D.R.Larson, A.R.Sourour [28]. The above papers gave rise to a series of works devoted to the description of mappings which are close to automorphisms and derivations of C^*-algebras and operator algebras. R.V.Kadison set out a program of study for local maps in [25], suggesting that local derivations could prove useful in building derivations with particular properties. R.V.Kadison proved in [25, Theorem A] that each continuous local derivation of a von Neumann algebra M into a dual Banach M-bimodule is a derivation. This theorem gave way to studies on derivations on C^*-algebras, culminating with a result due to B.E.Johnson, which asserts that every local derivation of a C^*-algebra A into a Banach A-bimodule is automatically continuous, and hence is a derivation [24, Theorem 5.3].

Let us present a list of finite or infinite dimensional algebras for which all local derivations are derivations: finite dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero [9]; Borel subalgebras of finite-dimensional simple Lie algebras [34]; infinite dimensional Witt algebras over an algebraically closed field of characteristic zero [19]; Witt algebras over a field of prime characteristic [32]; solvable Lie algebras of maximal rank [26]; Cayley algebras [6]; finite dimensional semi-simple Leibniz algebras over an algebraically closed field of characteristic zero [27]; locally finite split simple Lie algebras over a field of characteristic zero [13]; the Schrödinger algebras [4]; Lie superalgebra $q(n)$ [18]; conformal Galilei algebras [5].

On the other hand, some algebras (in most cases close to nilpotent algebras) admit pure local derivations, that is, local derivations which are not derivations. Below a short list of some classes of algebras which admit pure local derivations: finite dimensional filiform Lie algebras [9]; p-filiform Leibniz algebras [12]; solvable Leibniz algebras with abelian nilradicals, which have a one dimensionial complementary space [7]; the algebra of lower triangular $n \times n$-matrices [21]; the ternary Malcev algebra M_8 [22]; direct sum null-filiform Leibniz algebras [3].

The notions of 2-local derivations and 2-local automorphisms on algebras were introduced in 1997 by P.Šemrl [29]. The main problems concerning the above notions are to find conditions on the underlying algebra under which every 2-local derivation (respectively, 2-local automorphism) on this algebra automatically becomes a derivation (respectively, automorphism), and also to present examples of algebras with 2-local derivations (respectively, 2-local automorphism) that are not derivations (respectively, not automorphisms).

Let us present a list of finite or infinite dimensional algebras for which all 2-local derivations are derivations: finite dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero [8]; finite dimensional simple Leibniz algebras over an algebraically closed field of characteristic zero [11]; infinite dimensional Witt algebras over an algebraically closed field of characteristic zero [14]; locally finite split simple Lie algebras over a field of characteristic zero [13]; Virasoro algebras [16]; $W(2,2)$ Lie algebras [30]; Virasoro-like algebra [31]; the Schrödinger-Virasoro algebra [23]; Jacobson-Witt algebras [33]; deformative super W-algebras $W^s_\lambda(2,2)$ [1]; planar Galilean conformal algebra [17].

On the other hand, some algebras (in most cases close to nilpotent algebras) admit pure 2-local derivations, that is, 2-local derivations which are not derivations. Below a short list of some classes of algebras which admit pure 2-local derivations: finite dimensional nilpotent Lie algebras [8]; p-filiform Leibniz algebras [12]; solvable Leibniz algebras with abelian nilradicals, which have a one dimensionial complementary space [7]; finite dimensional nilpotent Leibniz algebras [11]; Thin Lie algebras [15, 16];

In the present paper we give description of local derivations on three and four-dimensional Zinbiel algebras. We also give a criterion of a linear operator on Zinbiel algebras of dimension three and four to be a local derivation. Moreover, similar problem concerning 2-local derivations on all three and four-dimensional Zinbiel algebras are investigated.

1 Preliminaries

Definition 1. *An algebra A over a field $\mathbb{F}$ is called a Zinbiel algebra if for any $x, y, z \in A$ the identity*

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y) \tag{1}$$

holds.

For an arbitrary Zinbiel algebra define the lower central series

$$A^1 = A, \quad A^{k+1} = A \circ A^k, \quad k \geq 1.$$

Definition 2. *A Zinbiel algebra A is called nilpotent if there exists an $s \in N$ such that $A^s = 0$. The minimal number s satisfying this property is called nilindex of the algebra A.*

Summarizing the results of [2], [20], and we give the classification of complex Zinbiel algebras dimension three and four.

Theorem 1. *An arbitrary non split Zinbiel algebra is isomorphic to the following pairwise non isomorphic algebras:*
$DimA = 3:$

$Z_3^1: \ e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = \frac{1}{2}e_3, \quad e_2 \circ e_1 = e_3;$

$Z_3^2: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3;$

$Z_3^3: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_2 = \alpha e_3, \quad \alpha \in C;$

$Z_3^4: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_3.$

$DimA = 4:$

$Z_4^1: \ e_1 \circ e_1 = e_2, \ e_1 \circ e_2 = e_3, \ e_2 \circ e_1 = 2e_3, \ e_1 \circ e_3 = e_4, \ e_2 \circ e_2 = 3e_4, \ e_3 \circ e_1 = 3e_4;$

$Z_4^2: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_4, \quad e_1 \circ e_3 = e_4, \quad e_3 \circ e_1 = 2e_4;$

$Z_4^3: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_2 = e_4, \quad e_3 \circ e_1 = 2e_4;$

$Z_4^4: \ e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3;$

$Z_4^5: \ e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = e_4;$

$Z_4^6: \ e_1 \circ e_1 = e_4, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = -2e_3 + e_4;$

$Z_4^7: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_4, \quad e_2 \circ e_2 = -e_3;$

$Z_4^8(\alpha): \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_4, \quad e_2 \circ e_1 = -\alpha e_3, \quad e_2 \circ e_2 = -e_4, \quad \alpha \in C$

$Z_4^9(\alpha): \ e_1 \circ e_1 = e_4, \ e_1 \circ e_2 = \alpha e_4, \ e_2 \circ e_1 = -\alpha e_4, \ e_2 \circ e_2 = e_4, \ e_3 \circ e_3 = e_4, \quad \alpha \in C$

$Z_4^{10}: \ e_1 \circ e_2 = e_4, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_4, \quad e_2 \circ e_2 = e_4, \quad e_3 \circ e_1 = e_4;$

$Z_4^{11}: \ e_1 \circ e_1 = e_4, \quad e_1 \circ e_2 = e_4, \quad e_2 \circ e_1 = -e_4, \quad e_3 \circ e_3 = e_4;$

$Z_4^{12}: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_4;$

$Z_4^{13}: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = e_4;$

$Z_4^{14}: \ e_2 \circ e_1 = e_4, \quad e_2 \circ e_2 = e_3;$

$Z_4^{15}(\alpha):\ e_1\circ e_2=e_4,\quad e_2\circ e_2=e_3,\quad e_2\circ e_1=\frac{1+\alpha}{1-\alpha}e_4,\quad \alpha\in C\setminus\{1\};$

$Z_4^{16}:\ e_1\circ e_2=e_4,\quad e_2\circ e_1=-e_4,\quad e_3\circ e_3=e_4;$

A derivation on a Zinbiel algebra $\mathcal{L}$ is a linear map $D:\mathcal{L}\to\mathcal{L}$ which satisfies the Leibniz rule:

$$D([x,y])=[D(x),y]+[x,D(y)],\quad \text{for any}\quad x,y\in\mathcal{L}. \tag{2}$$

The set of all derivations of $\mathcal{L}$ is denoted by $\mathrm{Der}(\mathcal{L})$ and with respect to the commutation operation is a Lie algebra.

Definition 3. *A linear operator Δ is called a local derivation if for any $x\in\mathcal{L}$, there exists a derivation $D_x:\mathcal{L}\to\mathcal{L}$ (depending on x) such that $\Delta(x)=D_x(x)$.*

Definition 4. *A map $\nabla:\mathcal{L}\to\mathcal{L}$ (not necessary linear) is called 2-local derivation if for any $x,y\in\mathcal{L}$ there exists a derivation $D_{x,y}\in Der(\mathcal{L})$ such that*

$$\nabla(x)=D_{x,y}(x),\quad \nabla(y)=D_{x,y}(y).$$

The set of all 2-local derivations on $\mathcal{L}$ we denote by $TLocDer(\mathcal{L})$.
For a 2-local derivation ∇ on $\mathcal{L}$ and $k\in\mathbb{C}$, $x\in\mathcal{L}$, we have

$$\nabla(kx)=D_{x,kx}(kx)=kD_{x,kx}(x)=k\nabla(x).$$

The following theorems describes derivations on nilpotent Zinbiel algebras of small dimension.

Theorem 2. *The derivations of 3-dimensional nilpotent Zinbiel algebras are given as follows:*

- *for the algebra Z_3^1 :*

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1+a_{2,1}e_2+a_{3,1}e_3,\\ D(e_2) &=& 2a_{1,1}e_2+\frac{3}{2}a_{2,1}e_3,\\ D(e_3) &=& 3a_{1,1}e_3. \end{array}$$

- *for the algebra Z_3^2 :*

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1+a_{2,1}e_2+a_{3,1}e_3,\\ D(e_2) &=& a_{1,2}e_1+a_{2,2}e_2+a_{3,2}e_3,\\ D(e_3) &=& (a_{1,1}+a_{2,2})e_3. \end{array}$$

- *for the algebra $Z_3^3(\alpha)$:*

$$\begin{array}{rcl} D(e_1) &=& \frac{1}{2}(-a_{2,1}+a_{3,3})e_1+a_{2,1}e_2+a_{3,1}e_3,\\ D(e_2) &=& \alpha a_{2,1}e_1+\frac{1}{2}(a_{2,1}+a_{3,3})e_2+a_{3,2}e_3,\\ D(e_3) &=& a_{3,3}e_3. \end{array}$$

- *for the algebra* Z_3^4 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1 + a_{2,1}e_2 + a_{3,1}e_3, \\ D(e_2) & = & (a_{1,1} + 2a_{2,1})e_2 + a_{3,2}e_3, \\ D(e_3) & = & 2(a_{1,1} + a_{2,1})e_3. \end{array}$$

Proof. The proof is carrying out by straightforward verification of derivation property 2. □

Theorem 3. *The derivations of 4-dimensional nilpotent Zinbiel algebras algebras are given as follows:*

- *for the algebra* Z_4^1 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1, + a_{2,1}e_2 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) & = & 2a_{1,1}e_2 + 3a_{2,1}e_3 + 4a_{3,1}e_4, \\ D(e_3) & = & 3a_{1,1}e_3 + 6a_{2,1}e_4, \\ D(e_4) & = & 4a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^2 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1 + a_{2,1}e_2 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) & = & 2a_{1,1}e_2 + a_{4,2}e_4, \\ D(e_3) & = & 2a_{1,1}e_3 + (a_{2,1} + 3a_{3,1})e_4, \\ D(e_4) & = & 3a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^3 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) & = & \frac{3}{2}a_{1,1}e_2 + a_{4,2}e_4, \\ D(e_3) & = & 2a_{1,1}e_3 + 3a_{3,1}e_4, \\ D(e_4) & = & 3a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^4 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) & = & a_{2,2}e_2 + a_{4,2}e_4, \\ D(e_3) & = & (a_{1,1} + a_{2,2})e_3, \\ D(e_4) & = & (2a_{1,1} + a_{2,2})e_4. \end{array}$$

- *for the algebra* Z_4^5 :

$$\begin{array}{rcl} D(e_1) & = & a_{1,1}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) & = & 2a_{1,1}e_2 - 2a_{2,1}e_3 + a_{4,2}e_4, \\ D(e_3) & = & 3a_{1,1}e_3 - a_{2,1}e_4, \\ D(e_4) & = & 4a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^6 :

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &=& a_{1,1}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &=& 2a_{1,1}e_3, \\ D(e_4) &=& 2a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^7 :

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &=& a_{1,1}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &=& 2a_{1,1}e_3, \\ D(e_4) &=& 2a_{1,1}e_4. \end{array}$$

- *for the algebra* $Z_4^8(\alpha)$:

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &=& a_{1,1}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &=& 2a_{1,1}e_3, \\ D(e_4) &=& 2a_{1,1}e_4. \end{array}$$

- *for the algebra* $Z_4^9(\alpha)$:

$$\begin{array}{rcl} D(e_1) &=& \frac{1}{2}a_{4,4}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) &=& -a_{2,1}e_1 + \frac{1}{2}a_{4,4}e_2 + a_{4,2}e_4, \\ D(e_3) &=& \frac{1}{2}a_{4,4}e_3 + a_{4,3}e_4, \\ D(e_4) &=& a_{4,4}e_4. \end{array}$$

- *for the algebra* Z_4^{10} :

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) &=& a_{1,1}e_2 - a_{2,1}e_3 + a_{4,2}e_4, \\ D(e_3) &=& a_{1,1}e_3 + a_{4,3}e_4, \\ D(e_4) &=& 2a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^{11} :

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) &=& a_{1,1}e_2 + a_{4,2}e_4, \\ D(e_3) &=& a_{1,1}e_3 + a_{4,3}e_4, \\ D(e_4) &=& 2a_{1,1}e_4. \end{array}$$

- *for the algebra* Z_4^{12} :

$$\begin{array}{rcl} D(e_1) &=& a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &=& a_{2,2}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &=& (a_{1,1} + a_{2,2})e_3, \\ D(e_4) &=& (a_{1,1} + a_{2,2})e_4. \end{array}$$

- *for the algebra* Z_4^{13} :

$$\begin{aligned} D(e_1) &= a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &= a_{1,2}e_1 + a_{2,2}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &= (a_{1,1} + a_{2,2})e_3, \\ D(e_4) &= 2a_{2,2}e_4. \end{aligned}$$

- *for the algebra* Z_4^{14} :

$$\begin{aligned} D(e_1) &= a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &= a_{1,2}e_1 + a_{2,2}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &= 2a_{2,2}e_3 + a_{1,2}e_4, \\ D(e_4) &= (a_{1,1} + a_{2,2})e_4. \end{aligned}$$

- *for the algebra* $Z_4^{15}(\alpha)$:

$$\begin{aligned} D(e_1) &= a_{1,1}e_1 + a_{3,1}e_3 + a_{4,1}e_4, \\ D(e_2) &= a_{1,2}e_1 + a_{2,2}e_2 + a_{3,2}e_3 + a_{4,2}e_4, \\ D(e_3) &= 2a_{2,2}e_3 + \tfrac{2}{1-\alpha}a_{1,2}e_4, \\ D(e_4) &= (a_{1,1} + a_{2,2})e_4. \end{aligned}$$

- *for the algebra* Z_4^{16} :

$$\begin{aligned} D(e_1) &= a_{1,1}e_1 + a_{2,1}e_2 + a_{4,1}e_4, \\ D(e_2) &= a_{1,2}e_1 + a_{2,2}e_2 + a_{4,2}e_4, \\ D(e_3) &= \tfrac{1}{2}(a_{1,1} + a_{2,2})e_3 + a_{4,3}e_4, \\ D(e_4) &= (a_{1,1} + a_{2,2})e_4. \end{aligned}$$

Proof. The proof is carrying out by straightforward verification of derivation property 2. □

2 Local and 2-local derivation on Zinbiel algebras of small dimension

Now we study local and 2-local derivations on Zinbiel algebras of small dimension.

Theorem 4. *The local derivations of 3-dimensional Zinbiel algebras are given as follows:*

- *for the algebra* Z_3^1 :

$$\begin{aligned} \Delta(e_1) &= c_{1,1}e_1, +c_{2,1}e_2 + c_{3,1}e_3, \\ \Delta(e_2) &= c_{2,2}e_2 + c_{3,2}e_3, \\ \Delta(e_3) &= c_{3,3}e_3. \end{aligned}$$

- *for the algebra* Z_3^2 :

$$\begin{array}{rcl}\Delta(e_1) &=& c_{1,1}e_1 + c_{2,1}e_2 + c_{3,1}e_3,\\ \Delta(e_2) &=& c_{1,2}e_1 + c_{2,2}e_2 + c_{3,2}e_3,\\ \Delta(e_3) &=& c_{3,3}e_3.\end{array}$$

- *for the algebra* $Z_3^3(\alpha)$:

$$\begin{array}{rcl}\Delta(e_1) &=& c_{1,1}e_1 + c_{2,1}e_2 + c_{3,1}e_3,\\ \Delta(e_2) &=& \alpha c_{1,2}e_1 + c_{2,2}e_2 + c_{3,2}e_3,\\ \Delta(e_3) &=& c_{3,3}e_3.\end{array}$$

- *for the algebra* Z_3^4 :

$$\begin{array}{rcl}\Delta(e_1) &=& c_{1,1}e_1 + c_{2,1}e_2 + c_{3,1}e_3,\\ \Delta(e_2) &=& c_{2,2}e_2 + c_{3,2}e_3,\\ \Delta(e_3) &=& c_{3,3}e_3.\end{array}$$

Proof. Since the proof repeats the same arguments that were presented earlier for each case, a detailed proof will be given only for the algebra Z_3^1, the rest of the cases are similar.

Let $\mathfrak{C}$ be the matrix of Δ and let Δ be an arbitrary local derivation on Z_3^1 :

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3}\\ c_{2,1} & c_{2,2} & c_{2,3}\\ c_{3,1} & c_{3,2} & c_{3,3}\end{pmatrix}$$

By the definition for all $x = \sum\limits_{i=1}^{3} x_i e_i \in Z_3^1$ there exists a derivation D_x on Z_3^1 such that

$$\Delta(x) = D_x(x).$$

By Theorem 2, the operator D_x has the following matrix form:

$$\mathfrak{C}_x = \begin{pmatrix} a_{1,1}^x & 0 & 0\\ a_{2,1}^x & 2a_{1,1}^x & 0\\ a_{3,1}^x & \frac{3}{2}a_{2,1}^x & 3\alpha_{1,1}^x\end{pmatrix}$$

Let $\mathfrak{C}$ be the matrix of Δ then by choosing subsequently $x = e_1,\ x = e_2,\ x = e_3$ and using $\Delta(x) = D_x(x)$, i.e. $\mathfrak{C}\overline{x} = D_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to x, which implies

$$\mathfrak{C} = \begin{pmatrix} c_{1,1} & 0 & 0\\ c_{2,1} & c_{2,2} & 0\\ c_{3,1} & c_{3,2} & c_{3,3}\end{pmatrix}$$

Using again $\Delta(x) = D_x(x)$, i.e. $\mathfrak{C}\overline{x} = \mathfrak{C}_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to $x = \sum\limits_{i=1}^{3} x_i e_i$, we obtain the next system of equalities

$$\begin{array}{ll} c_{1,1}x_1 & = a^x_{1,1}x_1, \\ c_{2,1}x_1 + c_{2,2}x_2 & = a^x_{2,1}x_1 + 2a^x_{1,1}x_2, \\ c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 & = a^x_{3,1}x_1 + \frac{3}{2}a^x_{2,1}x_2 + 3a^x_{1,1}x_3. \end{array} \tag{3}$$

Let us consider the next cases:

Case 1: Let $x_1 \neq 0$, then from (3) we uniquely determine

$$a^x_{1,1} = c_{1,1}, \quad a^x_{2,1} = \frac{c_{2,1}x_1 + c_{2,2}x_2 - 2c_{1,1}x_2}{x_1},$$

$$a^x_{3,1} = \frac{c_{3,1}x_1 + c_{3,2}x_2 + c_{3,3}x_3 - \frac{3}{2}a^x_{2,1}x_2 - 3c_{1,1}x_3}{x_1}.$$

Case 2: Let $x_1 = 0$ and $x_2 \neq 0$, then from (3) we uniquely determine

$$a^x_{1,1} = \frac{c_{2,2}}{2}, \ a^x_{2,1} = \frac{2(c_{3,2}x_2 + c_{3,3}x_3) - 3c_{2,2}x_3}{3x_2}.$$

Case 3: Let $x_1 = x_2 = 0$ and $x_3 \neq 0$, then $a^x_{1,1} = \frac{c_{3,3}}{3}$.

□

By direct calculation we obtain the dimensions of the spaces of derivation and local derivations to 3-dimensional nilpotent Zinbiel algebras.

Algebra	The dimensions of the space of derivations	The dimensions of the space of local derivations
Z_3^1	3	6
Z_3^2	6	7
$Z_3^3(\alpha)$	4	7
Z_3^4	4	6

Corollary 1. *The 3-dimensional Zinbiel algebras admit local derivations which are not derivations.*

Theorem 5. *The local derivations of 4-dimensional Zinbiel algebras algebras are given as follows:*

- *for the algebra* Z_4^1 :

$$\begin{array}{lll} \Delta(e_1) & = & c_{1,1}e_1, + c_{2,1}e_2 + c_{3,1}e_3 + c_{4,1}e_4, \\ \Delta(e_2) & = & c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4, \\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4, \\ \Delta(e_4) & = & c_{4,4}e_4. \end{array}$$

- *for the algebra* Z_4^2 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{2,1}e_2 + c_{3,1}e_3 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3 + c_{4,3}e_4,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* Z_4^3 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3 + c_{4,3}e_4,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* Z_4^4 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* Z_4^5 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3 + c_{4,3}e_4,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* Z_4^6 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* Z_4^7 :

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* $Z_4^8(\alpha)$:

$$\begin{aligned}
\Delta(e_1) &= c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\
\Delta(e_2) &= c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\
\Delta(e_3) &= c_{3,3}e_3,\\
\Delta(e_4) &= c_{4,4}e_4.
\end{aligned}$$

- *for the algebra* $Z_4^9(\alpha)$:

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{1,2}e_1 + c_{2,2}e_2 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{10} :

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{2,2}e_2 + c_{2,1}e_3 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{11} :

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{2,2}e_2 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{12} :

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{13} :

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{1,2}e_1 + c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{14} :

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{1,2}e_1 + c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* $Z_4^{15}(\alpha)$:

$$\begin{array}{rcl}\Delta(e_1) & = & c_{1,1}e_1 + c_{3,1}e_3 + c_{4,1}e_4,\\ \Delta(e_2) & = & c_{1,2}e_1 + c_{2,2}e_2 + c_{3,2}e_3 + c_{4,2}e_4,\\ \Delta(e_3) & = & c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) & = & c_{4,4}e_4.\end{array}$$

- *for the algebra* Z_4^{16} :

$$\begin{aligned}\Delta(e_1) &= c_{1,1}e_1 + c_{2,1}e_2 + c_{4,1}e_4,\\ \Delta(e_2) &= c_{1,2}e_1 + c_{2,2}e_2 + c_{4,2}e_4,\\ \Delta(e_3) &= c_{3,3}e_3 + c_{4,3}e_4,\\ \Delta(e_4) &= c_{4,4}e_4.\end{aligned}$$

By direct calculation we obtain the dimensions of the spaces of derivation and local derivations to 4-dimensional nilpotent Zinbiel algebras.

Algebra	The dimensions of the space of derivations	The dimensions of the space of local derivations
Z_4^1	4	10
Z_4^2	5	9
Z_4^3	4	8
Z_4^4	5	7
Z_4^5	4	9
Z_4^6	5	8
Z_4^7	5	8
$Z_4^8(\alpha)$	5	8
$Z_4^9(\alpha)$	5	9
Z_4^{10}	5	9
Z_4^{11}	5	8
Z_4^{12}	6	8
Z_4^{13}	7	9
Z_4^{14}	7	10
$Z_4^{15}(\alpha)$	7	10
Z_4^{16}	7	9

Corollary 2. *The 4-dimensional nilpotent algebras admit local derivations which are not derivations.*

Now we investigate 2-local derivations on 3-dimensional and 4-dimensional Zinbiel algebras.

Theorem 6. *Any 2-local derivation of the algebra Z_3^1 is a derivation.*

Proof. Let ∇ be a 2-local derivation on Z_3^1, such that $\nabla(e_1) = 0$. Then for any element $x = \sum\limits_{i=1}^{3} x_i e_i \in Z_3^1$, there exists a derivation $D_{e_1,x}(x)$, such that

$$\nabla(e_1) = D_{e_1,x}(e_1), \quad \nabla(x) = D_{e_1,x}(x).$$

Hence,

$$0 = \nabla(e_1) = D_{e_1,x}(e_1) = a_{1,1}e_1, +a_{2,1}e_2 + a_{3,1}e_3,$$

which implies, $a_{1,1} = a_{2,1} = a_{3,1} = 0$.

Consequently, from the description of the derivation Z_3^1, we conclude that $D_{e_1,x} = 0$. Thus, we obtain that if $\nabla(e_1) = 0$, then $\nabla \equiv 0$.

Let now ∇ be an arbitrary 2-local derivation of Z_3^1. Take a derivation $D_{e_1,x}$, such that

$$\nabla(e_1) = D_{e_1,x}(e_1) \quad \text{and} \quad \nabla(x) = D_{e_1,x}(x).$$

Set $\nabla_1 = \nabla - D_{e_1,x}$. Then ∇_1 is a 2-local derivation, such that $\nabla_1(e_1) = 0$. Hence $\nabla_1(x) = 0$ for all $x \in Z_3^1$, which implies $\nabla = D_{e_1,x}$. Therefore, ∇ is a derivation. □

Theorem 7. *The 3-dimensional Zinbiel algebras Z_3^2, $Z_3^3(\alpha)$ and Z_3^4 admit 2-local derivations which are not derivations.*

Proof. Since the proof repeats the same arguments that were presented earlier for each case, a detailed proof will be given only for the algebra Z_3^2, the rest of the cases are similar.

Let us define a homogeneous non additive function f on $\mathbb{C}^2$ as follows

$$f(z_1, z_2) = \begin{cases} \frac{z_1^2}{z_2}, & \text{if } z_2 \neq 0, \\ 0, & \text{if } z_2 = 0. \end{cases}$$

where $(z_1, z_2) \in \mathbb{C}$. Consider the map $\nabla : Z_3^2 \to Z_3^2$ defined by the rule

$$\nabla(x) = f(x_1, x_2)e_3, \quad \text{where} \quad x = x_1e_1 + x_2e_2 + x_3e_3 \in Z_3^2.$$

Since f is not additive, ∇ is not a derivation.

Let us show that ∇ is a 2-local derivation. For the elements

$$x = x_1e_1 + x_2e_2 + x_3e_3, \quad y = y_1e_1 + y_2e_2 + y_3e_3,$$

we search a derivation D in the form:

$$D(e_1) = a_{3,1}e_3, \quad D(e_2) = a_{3,2}e_3, \quad D(e_3) = 0.$$

Assume that $\nabla(x) = D(x)$ and $\nabla(y) = D(y)$. Then we obtain the following system of equations for α_n and β :

$$\begin{cases} x_1a_{3,1} + x_2a_{3,2} = f(x_1, x_2), \\ y_1a_{3,1} + y_3a_{3,2} = f(y_1, y_2). \end{cases} \tag{4}$$

Case 1. Let $x_1y_2 - x_2y_1 = 0$, then the system has infinitely many solutions, because of right-hand side of system is homogeneous.

Case 2. Let $x_1y_2 - x_2y_1 \neq 0$, then the system has a unique solution. □

Theorem 8. *Any 2-local derivation of the algebra Z_4^1 is a derivation.*

Proof. The proof is similar to the proof of Theorem 8. □

Theorem 9. *The 4-dimensional Zinbiel algebras Z_4^2-Z_4^{16} admit 2-local derivations which are not derivations.*

Proof. The proof is similar to the proof of Theorem 7. □

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