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ALGORITHMIC CRITERION OF LOCALLY FINITE SEPARABILITY OF ALGEBRAS REPRESENTED OVER EQUIVALENCE $\alpha^2 \cup id\,\omega$

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Abstract

It has been established that for equivalences of the form $\alpha^2 \cup id\,\omega$, the locally finite separability of any universal algebra represented over it is equivalent to the immune of the complement α. It is shown that for finitely separable algebras this criterion does not meet.

Keywords: *enumerated algebras, morphism, representation of the universal algebra over the equivalence and η-algebra, characteristic transversal equivalences and enumerations, uniformly computable separate enumerations.*

Mathematics Subject Classification (2010): *03D45, 03C57,08A70.*

Introduction

For undefined basic concepts, see [1, 2, 3, 4]. Universal algebras of effective signatures are considered.

The notion of a computably separable algebra turned out to be useful for solving a number of problems both in the theory of computable algebras and in Computer Science (see the surveys [5, 6]).

For example, A.I. Maltsev showed in [3] that any positive enumeration of a finitely generated algebra with nonzero congruences of only finite index is computable, which raises the question of the validity of this assertion in the general case (without the condition of finite generation). It turned out that there are counterexamples, and it is precisely in the class of computably separable algebras ([7]).Another example is the Bergstra-Taker problem in the theory of abstract data types on the existence of an initial enrichment in a finitely based manifold for any finitely generated positive algebra ([8]). A negative solution to this problem was obtained by presenting the corresponding example of a finitely generated algebra having a computably separable positive numbering with an immune characteristic transversal ([9]).

The word equivalence is understood as an equivalence relation on the set of natural numbers ω. If η is an equivalence, then $\alpha \subseteq \omega$ is said to be η-closed if α is the union of suitable η-classes (that is, $x \in \alpha \wedge x = y \pmod{\eta} \Rightarrow y \in \alpha$).

If (A, ν) is an enumerated algebra, then its enumeration equivalence will be called, for brevity, the enumeration kernel and denoted by $ker(\nu)$. We will identify the concepts of enumeration and algorithmic representation of a universal algebra. In the same way, the representability property of an algebra over η equivalence will be understood as being of equal volume to the property "to be an η-algebra".

The characteristic transversal of an equivalence η is the set of minimal representatives of all cosets of this equivalence, i.e. $tr(\eta) = \{x \mid \forall y(x = y \pmod{\eta} \Rightarrow x \leqslant y)\}$.

The characteristic transversal of a numbering (algorithmic representation) ν is the characteristic transversal of its kernel, i.e. $tr(ker(\nu)) = tr(\{\langle x, y\rangle \mid \nu x = \nu y\})$, which, for the sake of brevity, will often be denoted by $tr(\nu)$. The characteristic transversal is an important algorithmic attribute of both equivalence and enumeration. For example, it is obvious that for a negative (positive) equivalence, its characteristic transversal is computably enumerable (co-enumerable).

Let (A, ν) be an enumerated algebra. A subset B of an algebra A is called ν-computable (ν-enumerable, ν-co-enumerable) if the set $\nu^{-1}(B)$. If it is clear from the context which numbering is meant, then the subsets of the algebra will simply be called computable (enumerable, co-enumerable) without the prefix ν.

For a given algebra, the problem is the impact on it of various numberings (algorithmic representations) and relations between them, the founder of which is AI Maltsev ([3]). On the other hand, one can fix the equivalence η on the set ω of natural numbers and choose the class of all algebras that is enumerative to achieve the equivalence.Such an approach, in which not algebra is considered a priority, but its numbering, as a certain, figuratively speaking, "algorithmically given coordinate system", is in a sense dual with respect to the classical one ([5]), but within the framework of this approach it turned out to be convenient formulate and solve a range of problems.

Denote by $\eta(\alpha)$ the equivalence of $\alpha^2 \cup id\ \omega$ for any $\alpha \subseteq \omega$.

Let η be an 'equivalence on ω'.

Definition 1. *An algebra A is said to be representable over an equivalence η on ω (or an η-algebra) if there exists a numbering of this algebra with kernel η.*

Equivalences type $\eta(\alpha)$ have proven to be very useful. Thus, the above-mentioned counterexamples to the problems of A.I.Maltsev and Bergstra-Taker turned out to be $\eta(\alpha)$-algebras for suitable α.

A special place among computably separable algebras is occupied by negative algebras, because an enumerated algebra is computably separable if and only if it is approximated by negative algebras ([5]), i.e. An exceptional feature of computably separable algebras, in contrast to other enumerated systems, is their approximability by negative algebras.

Let us emphasize the following. If (A, ν) is an enumerated algebra and ν is computably separable, then (A, ν) is approximated by negative algebras and, if $tr(\nu)$ is immune, then the characteristic transversals of all negative homomorphic images of the numbered algebra (A, ν) are computably enumerable subsets of the immune set $tr(\nu)$.Therefore, all these images are finite and, consequently, the algebra A is residually finite. However, the question of the possibility of reversing this fact turned out to be much more subtle, and its solution required the introduction of a number of specific concepts.

We note the importance of the property of residual finiteness from the point of view of algorithmic questions of algebra, especially finitely defined in finitely based varieties, while more subtle algorithmic recognition methods are defined by finitely separable and locally finitely separable algebras ([3]).

Recall that a universal algebra A is said to be finitely separable if for any of its subalgebras A_0 and any element $a \in A \setminus A_0$ there exists a congruence of finite index such that the modulo of which the element a differs from all elements subalgebra A_0. In other words, in the language of homomorphisms, there is a homomorphism φ from the algebra A to a finite algebra such that $\varphi(a) \notin \varphi(A_0)$.Similarly, an algebra is said to be locally finitely separable if for any of its finitely generated subalgebras and any element outside this subalgebra there exists a congruence of finite index that distinguishes the given element from the subalgebra. Obviously, the class of finitely separated algebras is a proper subclass of the class of locally finitely separated algebras.

Proceeding from what has been said, in this paper we consider locally finitely separated universal algebras representable over equivalences of the $\eta(\alpha)$ type. The main result of the paper is the following algebraic-algorithmic characterization of the equivalence of $\eta(\alpha)$ to it in the language of representability of locally finitely separable algebras over it.

For any co-infinite $\alpha \subseteq \omega$, any universal algebra representable over an equivalence $\eta(\alpha)$ is locally finitely separable if and only if the characteristic transversal of this equivalence is immune.

An example of an equivalence of $\eta(\alpha)$ with an immune characteristic transversal is constructed such that some $\eta(\alpha)$-algebra is not finitely separable.

Thus, the algebraic concept of locally finite separability turned out to be equivalent to the algorithmic concept of immunity. For finite separability, these concepts turned out to be different.

Definition 2. *An enumerated algebra (A, ν) is said to be computably separable if every pair of its distinct elements is separated by a suitable ν-computable set.*

The following negative approximability theorem plays an important role in the theory of computably separable algebras:

Theorem 1 ([10])**.** *A numbered algebra is computably separable if and only if it is approximated by negative algebras.*

Thus, within the framework of the structural theory of computably separable algebras, negative algebras define a class of objects from which all computably separable algebras are constructed (as suitable subdirect products).

Recall ([1, 2]) that if (A, μ), (B, ν) are numbered algebras, then a homomorphism $\varphi : A \longrightarrow B$ is called a morphism if it is effective on numbers, i.e. there exists a computable function f such that $\varphi\mu = \nu f$. If $\mathfrak{B} = \{(B_i, \nu_i) | i \in I\}$ is a family of numbered algebras, then the numbered algebra (A, μ) is said to be approximated by $\mathfrak{B}$-algebras if for every pair of distinct elements $a_0, a_1 \in A$ there exists a morphism $\varphi_{\{a_0,a_1\})}$ from (A, μ) to a suitable $\mathfrak{B}$-algebra distinguishing these elements (that is, $\varphi_{\{a_0,a_1\}}(a_0) \neq \varphi_{\{a_0,a_1\}}(a_1)$).

Definition 3. *An enumerated algebra (A, ν) is said to be computably separable if every pair of its distinct elements is separated by a suitable ν-computable (ν-computably enumerable) set.*

An equivalence will be called infinite if the number of its cosets is infinite. Speaking about the extension of the equivalence η_0, we mean its extension, which is an equivalence relation, i.e. any binary relation η_1 such that $\eta_0 \subseteq \eta_1$ and η_1 is an equivalence.A computably separable equivalence that is not uniform will be called non-uniform, i.e. the application of the adjective "non-uniform" to equivalences that are not computably separable in the framework of our considerations is incorrect.

Recall the definitions of some equivalences from [1].

Let $\alpha \subseteq \omega$. Then:

1) $\eta^{\alpha} = \{\langle 2x, 2x+1\rangle | x \in \alpha\rangle\} \cup \{\langle 2x+1, 2x\rangle | x \in \alpha\ range\} \cup id\ \omega$;

2) $\eta_{\alpha} = \alpha^2 \cup id\ \omega$;

3) $\eta^*_{\alpha} = \{\langle x, y\rangle | \gamma_x \setminus \alpha = \gamma_y \setminus \alpha\}$, where γ is the canonical enumeration of finite sets (equivalent to $\eta^*_{\alpha} = \{\langle x, y\rangle | \gamma_x \bigtriangleup \gamma_y \subseteq \alpha\}$).

It is easy to check that all these equivalences are computably separable.

Note that in our notation $\eta_{\alpha} = \eta(\alpha)$.

Definition 4. *An equivalence is said to be effectively infinite if there exists an infinite computably enumerable set of natural numbers that are pairwise distinct modulo it.*

In other words, the equivalence of η is effectively infinite if and only if there exists a one-to-one computable embedding $id\ \omega$ in η (that is, for a suitable computable function f $x \neq y \Rightarrow f(x) \neq f(y)$). An infinite equivalence that is not effectively infinite will be called inefficiently infinite.

Let K_0 be the class of equivalences with immune characteristic transversals, K_1 the class of inefficiently infinite equivalences, and K_2 the class of equivalences whose characteristic transversals are hyperimmune. The following takes place

Proposition 1. $K_0 \supset K_1 \supset K_2$. *All inclusions are own.*

Proof. The inclusion relations are obvious. Let us show the strictness of these inclusions. Let α be a set with an immune non-hyperimmune complement. Then $\eta^*_{\alpha} \in K_0 \setminus K_1$ and $\eta(\alpha) \in K_1 \setminus K_2$. The proposition is proved. □

1 Criterion for locally finite separability

First of all, we prove a simple but important

Proposition 2. *Let A be an arbitrary universal algebra representable over a computably separable equivalence η with immune characteristic transversal $tr(\eta)$. Then A is residually finite.*

Proof. Let the algebra A have a computably separable numbering ν with immune characteristic transversal $tr(ker(\nu))$ and $ker(\nu) = \eta$. Let a, b be different elements of the algebra A. there exist a surjective homomorphism $\varphi_{a,b}$ from (A, ν) effective on ν-numbers onto some negative algebra (B, ξ) (in the signature of the algebra A), distinguishing these elements (i.e. $\varphi_{a,b}(a) \neq \varphi_{a,b}(b)$) and a computable function h such that $\varphi_{a,b}\mu = \xi h$. It is clear that the quotient algebra $A/\theta(a,b)$ (where $\theta(a,b)$ denotes the congruence, $\{\langle c,d\rangle | \varphi_{a,b}(c) = \varphi_{a,b}(d)\}$) is isomorphic to B.

Consider the numbered algebra $(A/\theta(a,b), \nu^*)$, where $\nu^* = \theta(a,b)\nu$, which is computably isomorphic to the negative algebra $(B,\ xi)$. In this case, the computable isomorphism is realized by the reducing function h. Because $\nu^*(x) = \nu^*(y) \Leftrightarrow \xi h(x) = \xi h(y)$ and ξ is negative, then so is ν^* also negative.Now note that $ker(tr(\nu^*)) \subseteq ker(tr(\nu))$, because if the equivalence η_0 is extended by the equivalence η_1, then the characteristic transversal of the extension η_1 is a subset of the characteristic transversal of the extended equivalence η_0. But $ker(tr(\nu)$ is immune, and $ker(tr(\nu^*))$ is computably enumerable (since ν^* is negative). Therefore, $ker(tr(\nu^*))$, and hence B is finite. Thus, A is finitely approximable. The proposition is proved. □

Theorem 2. *For any co-infinite $\alpha \subseteq \omega$ the following conditions are equivalent:*
(1) $tr(\eta(\alpha))$ – immune;
(2) every $\eta(\alpha)$-algebra is residually finite;
(3) every $\eta(\alpha)$-algebra is locally finitely separable.

Proof. (1)⇒(2) follows from Proposition 2.

(2)⇒(1). Let any $\eta(\alpha)$-algebra be residually finite. Assume that $tr(\eta(\alpha))$ is not immune and Fix an infinite computable subset $\beta = \{b_0, b_1, \dots\}$ and define $\eta(\alpha)$-unar $U = \langle \omega/\eta(\alpha); p\rangle$ as follows:

$$x \in \beta \wedge x = b_{n+1} \Rightarrow p(x) = b_n; p(b_0) = b_0; x \notin \beta \Rightarrow p(x) = x.$$

$\beta \subseteq \omega \setminus \alpha$ is an infinite computable set. Then the elements b_0 and b_1 do not differ by any congruence of finite index.

(3)⇒(1). Assume that every $\eta(\alpha)$-algebra is locally finitely separable, but $tr(\eta(\alpha))$ is not immune. As in the previous subsection, we construct a unar U and consider the subalgebra $\{\beta; p\}$ in it. Then the subalgebra $\{b_0\}; p$ and no element of β differ by any congruence of finite index. Contradiction.

(1)⇒(3) We say that a family of functions F is compatible with the equivalence η if for each function from F the equivalence η is a congruence.

We will be based on the following assertion, proved in [9].

Lemma 1 ([9])**.** *Let F be an effective family of computable functions consistent with $\eta(\alpha)$, where $\omega \setminus \alpha$ is immune. Then for any finite set $\gamma \subset \omega \setminus \alpha$ there exists a finite extension δ such that $\gamma \subset \delta \subset \omega \setminus \alpha$ and any function from F is consistent with the $\eta(\omega \setminus \delta)$ equivalence.*

We use Lemma 1 as follows.

Consider an arbitrary $\eta(\alpha)$-algebra A. Let A_0 be a finitely generated subalgebra of A and the element a does not lie in A_0, i.e. $a \notin A \setminus A_0$. Let's consider two cases.

1. $a = \alpha$. In this case A_0 must lie entirely in $\omega \setminus \alpha$. By virtue of being finitely generated, this subalgebra is computably enumerable and therefore finite (because $\omega \setminus \alpha$ is immune). Let the inverse image of A_0 (under the natural homomorphism given by the numbering $\nu : \omega \longrightarrow A$, where $\nu(x) = x/\eta(\alpha)$) be $\gamma \subset \omega \setminus \alpha$. By Lemma 2.2.1, there exists a finite extension $\delta \supset \gamma$ such that $\gamma \subset \delta \subset \omega \setminus \alpha$ and the equivalence $\eta(\omega \setminus \delta)$ is congruence of the finite index of the algebra $A = \langle \omega/\eta(\alpha); F\rangle$. Obviously, this congruence distinguishes the subalgebra A_0 and the element $a = \alpha$.

2. $a \in \omega \setminus \alpha$. By Lemma 2.2.1, there exists a finite $\delta \subset \omega \setminus \alpha$ such that $\{a\} \subset \delta$ and all functions in F are consistent with the equivalence $\eta(\omega\ setminus \delta)$. Then the image of a in the finite quotient algebra $\omega/\eta(\omega \setminus \delta)$ is not "glued" to any element, not only A_0, but even $A \setminus \{a\}$. The theorem is proved.

□

Note that part of the proof (1)⇒(3) holds for any infinite effective signatures. Only the existence of a finite set of generators is required.

Theorem 3. *There is an equivalence of $\eta(\alpha)$ with the immune characteristic transversal such that some $\eta(\alpha)$-algebra is not finitely separable.*

Proof. Let $\delta_0 = \{0\}, \delta_1 = \{1, 2\}, \delta_2 = \{3, 4, 5\}, \ldots$ be a strong sequence of pairwise disjoint finite sets in which the set δ_n consists of $n + 1$ consecutive natural numbers. We define a computable function f so that $f(x) = x + 1$ for $x \in \delta_t \wedge x \neq \max \delta_t$. If $x = \max \delta_t$, then we set $f(x) = x$.

We fix an arbitrary hypersimple set β with a regression complement, some recalculation of which can be traced by a partial computable regression function ψ (examples of such sets can be found, for example, in the book by H. Rogers [4]). Define the set

$$\beta^* = \{z \mid \exists n[(z \in \delta_n \wedge n \in \beta) \vee z = \max \delta_n]\}.$$

and the equivalence $\eta(\beta^*) = \{\langle x, y\rangle \mid x, y \in \beta^*\} \cup id\ \omega$, i.e. $\eta(\beta^*)$ has a unique non-trivial coset β^*.

Lemma 2. *The set β^* is hypersimple with regressive complement.*

Proof. Obviously, β^* is computably enumerable. Assume that $\omega \setminus \beta^*$ is not hyperimmune and $\sigma_0, \sigma_1, \ldots$ is a strong table for $\omega \setminus \beta^*$ (i.e. $sigma_0, \sigma_1, \ldots$ is a sequence of pairwise disjoint finite sets, enumerable by canonical indices, each of which contains an element from the complement β^*).For any set $\alpha \subseteq \omega$, by $\delta^{-1}\alpha$ we denote the complete δ-preimage of the set α for the mapping δ defined above. Let us construct the following sequence of finite sets $\sigma_0^*, \sigma_1^*, \ldots$ step by step.

Step 0.

$$\sigma_0^* = \bigcup_{x \in \sigma_0} \{z \mid x \in \delta_z\}.$$

Step $s + 1$.

$$\sigma_{s+1}^* = \bigcup_{x \in \sigma_k} \{z \mid x \in \delta_z\},$$

where

$$k = \min\{y \mid [\delta^{-1}\sigma_y \cap \delta^{-1}(\sigma_0^* \cup \dots \cup \sigma_s^*) = \emptyset]\}.$$

End of step $s+1$.

It is easy to show by induction on construction steps that each step ends with the presentation of the canonical index of a finite set that contains an element from the complement β and does not intersect with all sets constructed before this step, which contradicts the hyperimmunity of $\omega \setminus \beta$.

Let us show that $\omega \setminus \beta$ is regressive. To do this, we call an element of the set δ_n premaximal if it is not maximal, but greater than all other elements from δ_n. For convenience, we will assume that $\delta_0, \delta_1 \subseteq \beta^*$. We define a computable function ψ^* as follows. For a given natural x, find n such that $x \in \delta_n$ and set $\psi^*(x) = f(x)$ if x is not premaximal in δ_n . Otherwise, i.e. for premaximal x, we set $\psi^*(x) = \min \delta_{\psi(n)}$.It is easy to see that ψ^* regresses some recalculation of the set $\omega \setminus \beta^*$, because each element from $\omega \setminus \beta$ is "replaced" by a suitable segment δ_n and, in essence, these segments are regressed while preserving the correctness of the ψ^* operation. The lemma is proved. □

Let us show that over $\eta(\beta^*)$ a non-finitely separable algebra is representable.

We fix two elements $s, t \in (\omega \setminus \beta^*)$ and define a computable algebra $A = \langle \omega; h \rangle$, where

$$h(x, y) = \begin{cases} t, \text{ if } y \neq s, \\ f(x), \text{ if } y = s. \end{cases}$$

It is obvious that the function h is consistent with the equivalence $\eta(\beta^*)$ and therefore the positive factor algebra $\langle A/\eta(\beta^*); h\rangle$ of the computable algebra A by the congruence $\eta(\beta^*)$. Then $A_0 = \langle \omega/\eta(\beta^*) \setminus (\beta^* \cup \{s\}); h\rangle$ is a subalgebra of $\langle A/\eta(\beta^*); h\rangle$ obtained from the original quotient algebra by removing two elements, β^* and $\{s\}/\eta(\beta^*)$.Let us show that the subalgebra A_0 and the element β^* do not differ by any congruence of finite index. To do this, on the complement of the set β^*, we introduce the function $d : \omega \setminus \beta^* \longrightarrow \omega$, which we call the distance to β^* and set $d(x) = \min\{n \mid f^n(x) \in \beta^*\}$.

Since for every n the number of elements from the complement β^* whose distance to β^* is greater than this n is infinite, then every congruence of finite index contains a congruence class that contains the pair such elements, say a, b, that $d(a) < d(b)$.From the congruence of these elements and the fact that β^* is a fixed point under the action of the function f, we conclude that, modulo this congruence, the elements β^* and $f^{d(a)}(b)/\eta_{\beta^*}$ are equal, although $f^{d(a)}(b) \notin \beta^*$.Thus, any congruence of finite index identifies an element β^* with some element of the subalgebra A_0. The theorem is proved.

□

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