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DIRICHLET PROBLEM IN THE CLASS OF $A(Z)$–HARMONIC FUNCTIONS

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Abstract

This paper work is devoted to the study of the Dirichlet problem in the class of $A(z)$ −harmonic functions.

Keywords: *$A(z)$-analytic function, $A(z)$-harmonic function, Laplace operator, $A(z)$−barier.*

Mathematics Subject Classification (2010): *30C62, 30G30, 31A05.*

Introduction

This paper work is devoted to the study of the Dirichlet problem in the class of $A(z)$ −harmonic functions. Solution of the Beltrami equation

$$f_{\bar{z}}(z) = A(z) f_z(z) \tag{1}$$

is called $A(z)-$ analytic function. It is well-known, equation (1) is directly related to quasiconformal mappings. In generally assumed that $A(z)$ is measurable function and $|A(z)| \leq C < 1$ almost everywhere in the domain $D \subset \mathbb{C}$. The real part of the solution of equation (1) (i.e. $u(z) := \operatorname{Re} f(z)$) is called $A(z)$ −harmonic function. The work consists of an introduction and three paragraphs. In the first paragraph we give brief information on $A(z)-$ analytic, $A(z)-$ harmonic functions that will be used in subsequent studies of $A(z)$ −harmonic functions, introduce the operator $\Delta_A u$, which is an analogue of the well-known Laplace operator Δu, the functional properties of $A(z)$ −harmonic functions, the Poisson integral formula for $A(z)-$ harmonic functions, mean theorems and analogue of the Harnac's theorem. In the second section, we give the definition of a $A(z)$ −subharmonic function and some of its properties. For exaple maximum principle for $A(z)$ −subharmonic functions, family locally uniformly bounded $A(z)$ −subharmonic functions and etc. A method for solving the Dirichlet problem for the Laplace equation based on the properties of subharmonic functions. O.Perron [7] gave the initial presentation of the method, which was substantially developed by N.Wiener and M.V.Keldysh [3]. The third section is devoted to the study of the Perron method for the Dirichlet problem in the class of $A(z)$ −harmonic functions.

1 On the class of $A(z)$ −analytic and $A(z)$ −harmonic functions

Solutions to the Beltrami equation:

$$\bar{D}_A f(z) := \frac{\partial f(z)}{\partial \bar{z}} - A(z) \frac{\partial f(z)}{\partial z} = 0$$

is directly related to quasiconformal mappings. In the general case, with respect to the function $A(z)$, it is assumed that it is measurable and $|A(z)| \leq C < 1$ almost everywhere in the domain under consideration $D \subset \mathbb{C}$. In the literature, solutions to eq. (1) are usually called $A(z)-$analytic functions.

Theorem 1. *[1] For any measurable function on the complex plane of the $\mathbb{C}$ function,*

$$A(z): \ \|A\|_{\mathbb{C}} := \sup_{z\in\mathbb{C}} \{|A(z)|\} < 1$$

there exists a unique homeomorphic solution $\psi(z)$ of eq. (1) such that the ψ points remain fixed 0, 1, ∞.

The first part of section 1 is based on the fundamental work of A. Sadullaev and N. Zhabborov [9]. The most interesting is the case when $A(z)-$the antianalytic function, $\partial A = 0$, in a domain $D \subset \mathbb{C}$ such that $|A(z)| \leq C < 1, \ \ \forall z \in D$. Then, according to (1), the class $A(z)-$of analytic functions $f \in O_A(D)$ is characterized by the fact that $\bar{D}_A f = 0$. Since the anti-analytic function is infinitely smooth, then $O_A(D) \subset C^\infty(D)$ ([8, 9, **?**]). In this case, the following

Theorem 2. *(An analogue of the Cauchy theorem, see [9]) If $f \in O_A(D) \bigcap C\left(\bar{D}\right)$, where is $D \subset \mathbb{C}-$a domain with a rectifiable boundary, ∂D,then*

$$\int_{\partial D} f(z)\left(dz + A(z)\,d\bar{z}\right) = 0\ .$$

Let us now assume that the domain $D \subset \mathbb{C}$ is convex and $\xi \in D-$its fixed point. Consider the function

$$K(z,\xi) = \frac{1}{2\pi i}.\frac{1}{z-\xi+\overline{\int_{\gamma(\xi,z)} \bar{A}(\tau)\,d\tau}}, \tag{2}$$

where $\gamma(\xi, z)-$is a smooth curve connecting the points ξ, $z \in D$.

Theorem 3. *(Cauchy formula, see [9]). Let $D \subset \mathbb{C}$ is a convex domain and $G \subset D$ is arbitrary subdomain, with a piecewise smooth boundary ∂G, which lies compactly in D. Then for any function $f(z) \in O_A(G) \bigcap C\left(\bar{G}\right)$ the formula holds*

$$f(z) = \int_{\partial G} K(\xi,z) f(\xi)\left(d\xi + A(\xi)\,d\bar{\xi}\right), z \in G. \tag{3}$$

Theorem 4. *(see [4, 5, 10]). The real part of the $A(z)-$analytic function $f \in O_A(D)$ satisfies the equation in the domain D*

$$\Delta_A u := \frac{\partial}{\partial z}\left[\frac{1}{1-|A|^2}\left[(1+|A|^2)\frac{\partial u}{\partial \bar{z}} - 2A\frac{\partial u}{\partial z}\right]\right] + \frac{\partial}{\partial \bar{z}}\left[\frac{1}{1-|A|^2}\left[(1+|A|^2)\frac{\partial u}{\partial z} - 2\bar{A}\frac{\partial u}{\partial \bar{z}}\right]\right] = 0. \tag{4}$$

And vice versa, if $D-$ simply connected domain, $u \in C^2(D)$, $u : D \to R$ twice differentiable function satisfies differential equation (4), then there exists $f \in O_A(G)$: $u = \operatorname{Re} f$.

In connection with Theorem 4, it is natural to introduce the concept of $A(z)-$harmonic function as follows:

Definition 1. *Twice differentiable function $u \in C^2(D), u : D \to R$ is called $A(z)-$harmonic in the domain D, if everywhere in D function $u(z)$ satisfies differential equation (4). The class of $A(z)-$harmonic functions in the domain D is denoted by $h_A(D)$.*

Theorem 5. *(about mean value, see [5])). Let D is convex domain. If the function $u(z)$ is $A(z)-$harmonic in the lemniscate $L(z,R) = \{\xi \in D : |\psi(z,\xi)| < R\} \subset D$, then for any $r < R$ the following mean values hold*

$$u(z) = \frac{1}{2\pi r} \oint\limits_{|\psi(z,\xi)|=r} u(\xi) \left|d\xi + A(\xi)d\bar{\xi}\right|, \tag{5}$$

$$u(z) = \frac{1}{\pi r^2} \iint\limits_{|\psi(z,\xi)|\le r} u(\xi) \left(1 - |A(\xi)|^2\right) \frac{d\xi \wedge d\bar{\xi}}{2i}, \tag{6}$$

where $\psi(z,\xi) = z - \xi + \overline{\int\limits_{\gamma(z,\xi)} \bar{A}(\tau)\, d\tau}$.

For $A(z)-$harmonic functions, the following Dirichlet problem is naturally considered: Dirichlet problem. A bounded domain is $G \subset D$ given and a continuous function is given $\varphi(\xi)$ on the boundary ∂G. It is required to find a $A(z)-$function $u(z) \in h_A(G) \bigcap C(\bar{G}) : \ u|_{\partial G} = \varphi$ that is harmonic in the domain G and continuous on the closure $\overline{G}$. The well-known classical Poisson formula is the simplest and also the most important example of solving the Dirichlet problem in the class of harmonic functions. In the case where the domain G is a lemniscate $G = L(a,R)$, the following analogue takes place

Theorem 6. *(an analogue of the Poisson formula for $A(z)-$harmonic functions see [10]). If the function is $\varphi(\xi)$ continuous on the boundary of the lemniscate $L(a,R) \subset D$, then the function*

$$u(z) = \frac{1}{2\pi R} \oint\limits_{|\psi(a,\xi)|=R} \varphi(\xi) \frac{R^2 - |\psi(a,z)|^2}{|\psi(\xi,z)|^2} \left|d\xi + A(\xi)d\bar{\xi}\right| \tag{7}$$

is a solution of the Dirichlet problem in $L(a,R)$. For continuous functions, the following harmonicity criteria hold. Let's assume that a $D-$convex domain.

Theorem 7. *(see [5]). For a function, the $u(z) \in C(D)$ following statements are equivalent:*

1) $u \in h_A(D)$;

2) For any $z \in D$ and $L(z,r) \subset\subset D$ the following equality holds

$$u(z) = \frac{1}{2\pi r} \int\limits_{|\psi(\xi,z)|=r} u(\xi) \left|d\xi + A(\xi)\, d\bar{\xi}\right|;$$

3) For any $z \in D$ and $L(z,r) \subset\subset D$ the following equality holds

$$u\left(z\right)=\frac{1}{\pi r^{2}}\iint\limits_{\left|\psi\left(\xi,z\right)\right|\leq r}u\left(\xi\right)d\mu. \qquad (8)$$

Corollary 1. *(extremum principle). If the function $u \in h_A(D)$ reaches extremum inside the domain of domain G, then $u \equiv const$.*

Corollary 2. *Dirichlet problem*

$$\Delta_A u\left(z\right)=0,\ z\in G,\ \ u\in h_A\left(G\right)\bigcap C\left(\bar{G}\right),\ \ u|_{\partial G}=\varphi,\ \ \varphi\in C\left(\partial G\right)$$

has a unique solution.

Theorem 8. *(analogue of the Harnac's see [7]) . A monotone sequence of harmonic functions $u_j \in h_A(D)$ either uniformly (inside D) converges to infinity, or uniformly converges to some harmonic function $u \in h_A(D)$.*

2 Class of $A\left(z\right)-$subharmonic functions

In this section, we have devoted to the study of some properties of subharmonic functions.

Definition 2. *A function $u : D \to [-\infty;\infty)$ is said to be $A\left(z\right)-$subharmonic in a convex domain $G \subset \mathbb{C}$ if it satisfies the following two conditions:*

1) $u(z)$ is upper semicontinuous, i.e. $\forall z_0 \in G$ there is an inequality

$$\overline{\lim_{w\to z_0}}\, u(w)\leq u(z_0), \qquad (9)$$

(It follows that the function is bounded from above on any compact subset of the domain G);

2) for each point $\forall z_0 \in G$ there is a number $r(z_0) > 0$ such that $r < r(z_0)$ the inequality holds for all

$$u(z_0)\leq\frac{1}{2\pi r}\oint\limits_{\left|\psi(\xi,z_0)\right|=r}u\left(\xi\right)\left|d\xi+A(\xi)d\bar{\xi}\right|, \qquad (10)$$

where is the function

$$\psi\left(\xi,z_0\right)=\xi-z_0+\overline{\int\limits_{\gamma(\xi,z_0)}\bar{A}\left(\tau\right)d\tau}$$

for a convex domain $G \subset \mathbb{C}$ exists and has a unique zero at a point z_0 (see [9]).

A function $u : D \to [-\infty; \infty)$ is called $A(z)$ −subharmonic in an arbitrary domain D, if it is $A(z)$ −subharmonic in any convex subdomain $G \subset D$. The class of $A(z)$ −subharmonic functions in the domain D is denoted by $sh_A(D)$. In what follows, for convenience, the trivial function $u \equiv -\infty$ will also be included in $sh_A(D)$. Let us present some simple properties $A(z)$ −of subharmonic functions. The following 4 properties are directly obtained from the definition.

1) a linear combination of $A(z)$ −subharmonic functions with non-negative coefficients is a $A(z)$ −subharmonic function:

$$u_j \in sh_A(D), c_j \geq 0 \ (j = 1, 2, ..., m) \Rightarrow c_1 u_1 + ... + c_m u_m \in sh_A(D);$$

2) the maximum of a finite number of $A(z)$ −subharmonic functions is also $A(z)$ −subharmonic:

$$u_j \in sh_A(D), (j = 1, 2, ..., m) \Rightarrow u(z) := \max\{u_1(z), ..., u_m(z)\} \in sh_A(D);$$

3) the limit of a monotonically decreasing sequence $A(z)$ −of subharmonic functions is $A(z)$ −subharmonic:

$$u_j \in sh_A(D), u_j(z) \geq u_{j+1}(z) \Rightarrow u(z) := \lim_{j\to\infty} u_j(z) \in sh_A(D);$$

4) a uniformly descending sequence of $A(z)$ −subharmonic functions converges to a subharmonic function:

$$u_j \in sh_A(D), u_j \rightrightarrows u \Rightarrow u \in sh_A(D);$$

Let us prove further properties

5) (maximum principle, see [6]). Let be $u \in sh_A(D)$ and at some point $z_0 \in D$ it reaches its maximum, then $u|_D \equiv const$.

Proof. Let $\exists z_0 \in D : u(z_0) = \sup_{z\in D} \{u(z)\}$. Consider the set

$$M := \{z \in D : u(z) = u(z_0)\}.$$

Then $z_0 \in M$and from the semicontinuity $u(z)$ in D a bunch of Mclosed in D. From the definition of a $A(z)$ −subharmonic function for an arbitrary fixed $w \in M$, we have

$$u(z_0) = u(w) \leq \frac{1}{2\pi r} \oint\limits_{|\psi(\xi,w)|=r} u(\xi) \left|d\xi + A(\xi)d\bar{\xi}\right| \leq u(z_0), \forall r \leq r(z_0).$$

Hence it follows that $u|_{\partial L(w,r)} \equiv u(z_0)$, because if $\exists \xi \in \partial L(p,r) : u(\xi) < u(z_0)$, then from semi-continuity $u(z) < u(z_0)$ in some non-empty open piece $\lambda \subset \partial L(w,r)$, which would contradict the equality $u(w) = u(z_0)$. And so

$$u|_{\partial L(w,r)} \equiv u(z_0), \forall 0 < r \leq r\left(z^0\right)$$

and $u|_{L(w,r)} \equiv u(z_0)$. Hence, $w \in M$is an interior point and $M$is an open set in D. Mean $M = D$. Property 5 is proved. □

6) if for functions $v \in sh_A(D)$, $u \in h_A(D)$ their narrowing on the border of the domain $G \subset\subset D$ satisfy the inequality $v|_{\partial G} \leq u|_{\partial G}$, then the inequality holds $v|_G \leq u|_G$. The proof simply follows from the maximum principle applied to the difference $u - v$. 7) if at the border ∂D given a continuous function $\varphi \in C(\partial D)$,then in the class

$$U = \{v \in sh_A(D) \cap C(D) : u|_{\partial D} \equiv \varphi\}$$

$A(z)$ −harmonic function $u : u|_{\partial D} = \varphi$ satisfies the maximum condition, i.e.

$$u(z) \geq v(z), \forall z \in D, \forall v \in U.$$

8) For $A(z)$ −a subharmonic function $v \in sh_A(D)$, where $D \subset \mathbb{C}$ is a convex domain, the second condition (10), which was originally required for sufficiently small $r < r(z_0)$, is satisfied for all $r : L(z_0, r) \subset\subset D$.

Proof. The existence of a decreasing sequence follows $\varphi_j \in C(\partial L(z_0, r)) : \varphi_j \downarrow v|_{\partial L(z_0,r)}$ from the upper semicontinuity at the boundary of the lemniscate $\partial L(z_0, r)$. Let us construct $A(z)-$ harmonic function u_j in the lemniscate $L(z_0, r)$ with the boundary value $u_j|_{\partial L(z_0,r)} = \varphi_j$. Since $\varphi_j \geq v|_{\partial L(z_0,r)}$, then according to property 6 implies that

$$u_j|_{L(z_0,r)} \geq v|_{L(z_0,r)}.$$

We have

$$v(z_0) \leq u_j(z_0) = \frac{1}{2\pi r} \oint\limits_{|\psi(z_0,\xi)|=r} u_j(\xi) \left|d\xi + A(\xi)\, d\bar{\xi}\right| = \frac{1}{2\pi r} \oint\limits_{|\psi(z_0,\xi)|=r} \varphi_j(\xi) \left|d\xi + A(\xi)\, d\bar{\xi}\right|$$

which $j \to \infty$ gives us

$$v(z_0) \leq \frac{1}{2\pi r} \oint\limits_{|\psi(z_0,\xi)|=r} v(\xi) \left|d\xi + A(\xi)\, d\bar{\xi}\right|.$$

The preposition is proved. □

If

$$\{u_1(z), u_2(z), ..., u_N(z)\} \subset sh_A(D)$$

is a finite family of $A(z)$ −subharmonic in $D \subset \mathbb{C}$ functions, then

$$u(z) = \sup\{u_1(z), u_2(z), ..., u_N(z)\}$$

is a $A(z)$ −subharmonic function. The situation is different when we consider an arbitrary family of $\{u_\alpha(z)\}$, $\alpha \in \Lambda$ subharmonic functions. In this case, it is necessary to require $u(z) = \sup\limits_{\alpha \in \Lambda} \{u_\alpha(z)\}$ was locally bounded, which is the same $\{u_\alpha(z)\}$ locally uniformly bounded.

Theorem 9. *Let* $\{u_\alpha(z)\},\ \alpha \in \Lambda$, *be an arbitrary locally uniformly bounded family of* $A(z)-$*subharmonic functions and* $u(z) = \sup\limits_{\alpha\in\Lambda}\{u_\alpha(z)\}$. *Then the regularization*

$$u^*(z) := \overline{\lim_{w\to z}}\, u(w)$$

is an $A(z)-$*subharmonic function in* D. *Further, if* $\{u_j(z)\}$ *is a sequence of locally uniformly bounded* $A(z)-$*subharmonic functions and*

$$u(z) = \overline{\lim_{j\to\infty}}\, u_j(z),$$

then $u^*(z)$ *is an* $A(z)-$*subharmonic function.*

Proof. We fix an arbitrary convex domain $G \subset D$ and, as usual, construct a function $\psi(z,\xi)$ in G. For each function $u_\alpha(z),\ \alpha \in \Lambda$, from the definition of a $A(z)-$subharmonic function we have

$$u_\alpha(z) \le \frac{1}{2\pi r}\int\limits_{|\psi(z,\xi)|=r} u_\alpha(\xi)\left|d\xi + A(\xi)\,d\bar{\xi}\right| = \left[\begin{cases} \psi(a,\xi) = \psi(a,z) + \psi(z,\xi) \\ \psi(z) := \psi(a,z) \\ \xi = z + \psi^{-1}(z,\xi) \\ = \end{cases}\right]$$

$$= \frac{1}{2\pi r}\int\limits_{|\psi(a,z)|=r} u_\alpha\left(z + \psi^{-1}(z,\xi)\right)\left|d\xi + A(\xi)\,d\bar{\xi}\right|,$$

where $r < r_0 : L(z, r_0) \subset\subset G$. Hence, $u(z) \le \frac{1}{2\pi r}\int\limits_{|\psi(a,z)|=r} u_\alpha\left(z + \psi^{-1}(z,\xi)\right)\left|d\xi + A(\xi)\,d\bar{\xi}\right|$.
Let us write this inequality for any $w \in L(z,\delta)$, where $\delta > 0$ such that $r + \delta < r_0$:

$$u(w) \le \frac{1}{2\pi r}\int\limits_{|\psi(a,z)|=r} u_\alpha\left(w + \psi^{-1}(z,\xi)\right)\left|d\xi + A(\xi)\,d\bar{\xi}\right|.$$

If we take a regularization on both sides, then

$$\overline{\lim_{w\to z}}\, u(z) \le \frac{1}{2\pi r}\int\limits_{|\psi(a,z)|=r} \overline{\lim_{w\to z}}\, u_\alpha\left(w + \psi^{-1}(z,\xi)\right)\left|d\xi + A(\xi)\,d\bar{\xi}\right|.$$

And we'll get that

$$u^*(z) \le \frac{1}{2\pi r}\int\limits_{|\psi(a,z)|=r} u^*\left(z + \psi^{-1}(z,\xi)\right)\left|d\xi + A(\xi)\,d\bar{\xi}\right| = \frac{1}{2\pi r}\int\limits_{|\psi(z,\xi)|=r} u^*(\xi)\left|d\xi + A(\xi)\,d\bar{\xi}\right|,$$

which proves the $A(z)-$subharmonicity u^* in G and since $G \subset D-$arbitrary, then u^* is $A(z)-$subharmonic in D. The second part of the Theorem, for a sequence of locally uniformly bounded $A(z)-$subharmonic functions, the proof is similar to the previous one. The theorem is proved. □

3 Perron method

Let us be given a bounded domain $D = \left\{ z \in \mathbb{C} : |A(z)| \le 1, \sup\limits_{z} |A(z)| \le 1 \right\} \subset \mathbb{C}$ and $G \subset\subset D$ a function $\varphi \in C(\partial G)$. The classical internal Dirichlet problem is that find function $\omega \in h_A(D) \cap C(D)$, $\omega|_{\partial D} = \varphi|_{\partial D}$. From the maximum principle for $A(z)-$harmonic functions, it immediately follows that if a solution to the Dirichlet problem exists, then it is unique. In one particular case, when the domain$L(z_0, r) \subset\subset G$ lemniscate, in the section 1 the solution was constructed constructively, explicitly by the Poisson integral. To solve the Dirichlet problem in a convex domain $G \subset \mathbb{C}$ we use the well-known Perron method. We consider it a very convenient apparatus in potential theory and in the theory of harmonic functions; perhaps the method is very useful in other boundary value problems of elliptic equations. For a given continuous function, $\varphi \in C(\partial G)$ we set

$$U_A(\varphi, G) = \left\{ u \in sh_A(G) : \overline{\lim_{z \to \xi \in \partial D}} u(\xi) \le \varphi(\xi) \right\}, \ \omega(z) := \sup\{u(z) : u \in U_A(\varphi, G)\}.$$

Theorem 10. *Function $\omega(z) \in h_A(G)$ and it coincides with its regularization i.e.*

$$\omega(z) = \omega^*(z), \forall z \in G.$$

Proof. As $\varphi \in C(\partial G)$ and the norm $\|\varphi\|_{\partial G}$ is bounded by a constant $M > 0$, then by the maximum principle, each function $u \in U_A(\varphi, G)$ bounded from above $u(z) \le M$ therefore, $\omega^*|_G \le M$. According to an analog of the Choquet Lemma, there exists a countable family of functions

$$u_j \in U_A(\varphi, G) : \ u^*(z) = (\sup\{u_j(z) : j \in \mathbb{N}\})^* = \omega^*(z).$$

The sequence $\omega_j(z) = \max\{u_1(z), u_2(z), ..., u_j(z)\}$ is an increasing sequence of $A-$subharmonic functions, and $\omega_j \in U_A(\varphi, G)$, $\omega_{j+1}(z) \ge \omega_j(z)$,$\omega_j(z) \underset{j \to \infty}{\to} \omega(z)$. We fix $L(z_0, r) \subset\subset G$ and $A(z)-$harmonize in $L(z_0, r)$, i.e.

$$\tilde{\omega}_j(z) = \begin{cases} \int\limits_{|\psi(\xi, z_0)| = r} w_j(\xi) \frac{R^2 - |\psi(z, z_0)|^2}{|\psi(\xi, z)|^2} \left| d\xi + A(\xi) d\bar{\xi} \right|, z \in L(z_0, r) \\ \omega_j(z), z \in G \backslash L(z_0, r) \end{cases}.$$

Then we find

$$\tilde{\omega}_j \in sh_A(G) \cap h_A(L(z_0, r))$$

such that it follows that

$$\tilde{\omega}_j(z) \ge \omega_j(z), \ \forall z \in L(z_0, r)$$

and

$$\tilde{\omega}_j(z) \uparrow \omega_j(z), \forall z \in G \backslash L(z_0, r).$$

Since, in addition $u(z) \leq M$, then by analogy with Harnack's theorem (see theorem 8) $u \in h_A(L(z_0, r))$. Therefore $u \in h_A(G)$, because $L(z_0, r) \subset\subset G-$arbitrary. The obvious inequality $u(z) \leq \omega(z)$ also $\omega|_G = \omega^*|_G$ implies

$$\omega^*(z) = u^*(z) \leq \omega(z) \leq \omega^*(z), \forall z \in G.$$

The theorem is proved . □

Function coincidence $\omega(z)$ at the boundary ∂G with a given function $\varphi(\xi)$ depends on the property ∂G,on the regularity of the domain G.

Definition 3. *We say that a domain $G \subset \mathbb{C}$ has a global $A(z)-$barrier at $\xi_0 \in \partial G$ if it exists $b \in sh_A(G)$ such that*

$$\lim_{\substack{z \to \xi_0 \\ z \in G}} b(z) = 0, \ \sup\{b(z) : z \in G \backslash L(\xi_0, r)\} < 0, \forall r > 0 : L(\xi_0, r) \subset G.$$

Domain $G \subset \mathbb{C}$ called has local $A(z)-$barrier at a point $\xi_0 \in \partial G$ if there exists a lemniscate $L(\xi_0, r) \subset D$ such that the intersection $L(\xi_0, r) \cap G$ has global barrier.

Proposition 1. *If the domain G has a local barrier at the point $\xi_0 \in \partial G$ then the domain G has a $A(z)-$global barrier at that point.*

Proof. $\exists L(\xi_0, R) \subset D$ such that $G \cap L(\xi_0, R)$ has a global $A(z)-$barrier at ξ_0, i.e. $\exists a \in sh_A(G \cap L(\xi_0, r)) : \lim\limits_{\substack{z \to \xi_0 \\ z \in G \cap L(\xi_0, r)}} a(z) = 0$ and

$$sup\{a(z) : z \in G, \ r < |\psi(\xi_0, z)| < R\} < 0, \forall r < R.$$

We fix $\delta \in (0; r)$ and consider following $A(z)-$subharmonic function in D: $l(z) = \frac{1}{2}|\sup\{b(z) : z \in G, \ \delta < |\psi(\xi_0, z)| < R\}| \frac{\ln \frac{|\psi(z, \xi_0)|}{R}}{\ln \frac{R}{\delta}}$. Then $l|_{\partial L(\xi_0, \delta)} > a|_{\partial L(\xi_0, \delta)}$. As

$$\lim_{\substack{z \to \xi_0 \\ z \in G \cap L(\xi_0, r)}} a(z) = 0$$

and $l(\xi_0) = -\infty$, then there exists $0 < \varepsilon < \delta$ such that $l|_{\partial L(\xi_0, \varepsilon)} < a|_{\partial L(\xi_0, \varepsilon)}$. Then it is easy to check that the function $b(z) = \begin{cases} a(z), z \in G \cap L(\xi_0, \delta) \\ \max\{a(z), l(z)\}, \ z \in G \cap \{L(\xi_0, \varepsilon) \backslash L(\xi_0, \delta)\} \\ l(z), z \in G \backslash L(\xi_0, \varepsilon) \end{cases}$ is $A(z)-$barrier at the point ξ_0. The preposition is proved . □

It follows that the local $A(z)-$barrier and the global $A(z)-$barrier are equivalent.

Theorem 11. *If the domain G has a point $\xi_0 \in \partial G$ $A(z)-$barrier, then*

$$\lim_{\substack{z \to \xi_0 \\ z \in G}} \omega(z) = \varphi(\xi_0).$$

Proof. Set $M = \|\varphi\|_{\partial D}$ and fix $\varepsilon > 0$.From continuity $\varphi \in C(\partial G)$, there exists $\delta > 0 : |\varphi(\xi) - \varphi(\xi_0)| < \varepsilon,\ \xi \in \partial G \cap L(\xi_0, \delta)$. Since at the point $\xi_0 \in \partial G$ there is a $A-$barrier, then there $b \in sh_A(G)$ is one such that

$$\lim_{\substack{z\to\xi_0\\ z\in G}} b(z) = 0,\ \sup\{b(z) : z \in G\backslash L(\xi_0, \varepsilon)\} = \lambda(\varepsilon) < 0.$$

Let us estimate the boundary values of the function

$$v_\varepsilon(z) = \varphi(\xi_0) - \varepsilon - \frac{b(z)}{\lambda(\varepsilon)}(M + \varphi(\xi_0)).$$

If $\xi \in \partial G \cap L(\xi_0, \delta)$, then

$$\overline{\lim_{\substack{z\to\xi\\ z\in D}}}\, v_\varepsilon(z) \leq -\varepsilon + \varphi(\xi_0) \leq \varphi(\xi).$$

If $\xi \in \partial G\backslash L(\xi_0, \delta)$, then

$$\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, v_\varepsilon(z) \leq -\varepsilon + \varphi(\xi_0) - M - \varphi(\xi_0) \leq \varphi(\xi).$$

Hence, $\overline{\lim\limits_{\substack{z\to\xi\\ z\in G}}}\, v_\varepsilon(z) \leq \varphi(\xi),\ \forall\xi \in \partial G$ and $v_\varepsilon \in U_A(\varphi, G)$. Hence $v_\varepsilon(z) \leq \omega(z)$ and

$$\underline{\lim_{\substack{z\to\xi_0\\ z\in G}}}\, \omega(z) \geq \underline{\lim_{\substack{z\to\xi_0\\ z\in G}}}\, v_\varepsilon(z) = -\varepsilon + \varphi(\xi_0),$$

which at $\varepsilon \to 0+$ gives

$$\underline{\lim_{\substack{z\to\xi_0\\ z\in G}}}\, \omega(z) \geq \varphi(\xi_0). \tag{11}$$

To prove the reverse inequality, we fix $u \in U_A(\varphi, G)$ and consider the sum

$$u(z) + w_\varepsilon(z) \in sh_A(G),$$

where $v_\varepsilon(z) = -\varphi(\xi_0) - \varepsilon - \frac{b(z)}{\lambda(\varepsilon)}(M - \varphi(\xi_0))$.

We have

$$\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, [u(z) + w_\varepsilon(z)] \leq \overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, u(z) + \overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, w_\varepsilon(z).$$

If $\xi \in \partial G \cap L(\xi_0, \delta)$, now then

$$\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, v_\varepsilon(z) \leq -\varepsilon - \varphi(\xi_0) \leq -\varphi(\xi).$$

If $\xi \in \partial G\backslash L(\xi_0, \delta)$, then

$$\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\, v_\varepsilon(z) \leq -\varepsilon - \varphi(\xi_0) - (M - \varphi(\xi_0)) \leq -\varepsilon - M \leq -\varphi(\xi).$$

From here

$$u + w_\varepsilon|_{\partial G} \le 0$$

and according to the principle of maximum

$$u + w_\varepsilon|_G \le 0,$$

i.e.

$$u|_G \le -w_\varepsilon|_G$$

. Since it is $u \in U_A(\varphi, G)$ −arbitrary, then

$$\omega|_G \le -w_\varepsilon|_G.$$

Hence, $\overline{\lim\limits_{\substack{z\to\xi_0\\ z\in G}}}\, \omega(z) \le \overline{\lim\limits_{\substack{z\to\xi_0\\ z\in G}}}\, [-w_\varepsilon(z)] = \varepsilon + \varphi(\xi_0)$ and when $\varepsilon \to 0+$ we get

$$\overline{\lim_{\substack{z\to\xi_0\\ z\in G}}}\, \omega(z) \le \varphi(\xi_0). \tag{12}$$

Combining this with (10) we arrive at $\lim\limits_{\substack{z\to\xi_0\\ z\in G}} \omega(z) = \varphi(\xi_0)$. The theorem is proved. □

Corollary 3. . *If the domain $G \subset D$ has a $A(z)$ −barrier at all boundary points $\xi \in \partial G$ then the Dirichlet problem of the equation*

$$\begin{cases} \Delta_A u = 0 \\ u|_{\partial G} = \varphi(\xi) \end{cases}$$

always (for any function $\forall\varphi \in C(\partial D)$) has a solution $u \in h_A(G) \cap C(\bar{G})$, and this solution is unique.

Definition 4. *A domain $G \subset D$ is called $A(z)$ −regular domain if it contains a negative $A(z)$ −subharmonic function $\rho \in sh_A(G)$ such that*

$$\rho|_G < 0, \ \lim_{z\to\xi\in\partial G} \rho(z) = 0.$$

Here the last condition means that for any number the $c < 0$ set $\{z \in G : \rho(z) \le c\}$ is a compact set in D.

The following theorem shows that there is a close relationship between $A(z)$ −regularity and $A(z)$ −barrier in domains.

Theorem 12. *The region G has a barrier at every point $\xi \in \partial G$if and only if domain G is $A(z)$ −regular.*

Proof. Let the domain G have a barrier at each point $\xi \in \partial G$. We fix a point $w \in G$ and a function $\frac{1}{2\pi} \ln |\psi(\xi, w)|$, $\xi \in \partial G$. According to the Dirichlet problem $\Delta_A u(z) = 0, \ u|_{\partial G} = \frac{1}{2\pi} \ln |\psi(\xi, w)|$ has a unique solution

$$u_w \in h_A(G) \cap C\left(\overline{G}\right).$$

Then $\frac{1}{2\pi} \ln |\psi(z, w)| - u_w(z)$ is the defining, $A(z)-$subharmonic exhaustion function of the domain G, i.e.

$$\frac{1}{2\pi} \ln |\psi(\xi, z)| - u_z(\xi) \in sh_A(G)$$

and

$$G = \left\{ z \in D : \frac{1}{2\pi} \ln |\psi(\xi, z)| - u_z(\xi) < 0 \right\}.$$

Let us G is an $A(z)-$regular and $\rho \in sh_A(G), \ \rho|_G < 0, \ \lim\limits_{z \to \xi \in \partial G} \rho(z) = 0$. We fix $\xi_0 \in \partial G$ and put $\varphi(\xi) = |\varphi(\xi, \xi_0)|^2$, $\xi \in \partial G$. Let's build a function $\omega(z)$. By Theorem 10, it is harmonic in G. Since the function $v(z) = |\psi(z, \xi_0)|^2, z \in G$ belongs to the class $U_A(\varphi, D)$, then $\omega(z) \geq |\psi(z, \xi_0)|^2$. Therefore, the function $b(z) = -\omega(z)$ satisfies the $\sup\{b(z) : z \in G \backslash L(\xi_0, r)\} < 0, \forall r > 0 : L(\xi_0, r) \subset G$ barrier function condition. It remains to prove the condition

$$\lim_{\substack{z \to \xi_0 \\ z \in G}} b(z) = 0.$$

Fixing the lemniscate $L = L(\xi_0, r) \subset D$ and compact $K \subset \partial L \cap G$. Then $\rho_0 := \max\limits_K \rho(z) < 0$. Let $M := \max\limits_{\partial G} |\psi(z, \xi_0)|^2$ and

$$\phi(\xi) := \begin{cases} M, z \in (\partial L \cap G) \backslash K \\ 0, z \in (\partial L \cap G) \cup K \end{cases}$$

We take the Poisson integral

$$u(z) := \int\limits_{\partial L} \phi(\xi) \Pi(\xi, z) \left| d\xi + A(\xi) d\bar{\xi} \right|, \forall z \in L.$$

Then $u \in h_A(L), 0 \leq u(z) \leq M$ and

$$u(\xi_0) = \frac{1}{2\pi r} \int\limits_{\partial L} \phi(\xi) \left| d\xi + A(\xi) d\bar{\xi} \right| = \frac{1}{2\pi r} \int\limits_{K'} \phi(\xi) \left| d\xi + A(\xi) d\bar{\xi} \right| = \frac{M \mu(K')}{2\pi r},$$

where $K' = (\partial L \cap G) \backslash K$, and $\mu(K') := \int\limits_{K'} \left| d\xi + A(\xi) d\bar{\xi} \right|$ is μ measure of the set K'. In addition, the boundary function $\phi(\xi) \equiv M$ on the open piece $K' = (\partial L \cap G) \backslash K$. Therefore, $u|_{K'} \equiv M$. We fix $w \in U_A(\varphi, G)$ and take the auxiliary $A(z)-$subharmonic into $L \cap G$ the function $f(z) = -r^2 + \frac{\rho(z)}{|\rho_0|} M - u(z)$. Let us show that $\overline{\lim\limits_{\substack{z\to\xi\\ z\in L\cap G}}}\left[w\left(z\right)+f\left(z\right)\right]\leq 0,\ \forall\xi\in\partial\left(L\cap G\right),$ from which it follows that $w\left(z\right)+f\left(z\right)\leq 0$ in $L\cap G$. At $\xi\in\bar{L}\cap\partial G$ we have

$$\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\left[w\left(z\right)+f\left(z\right)\right]\leq\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,w\left(z\right)+\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,f\left(z\right)\leq\phi\left(\xi\right)-r^2-u\left(\xi\right)=\left|\psi\left(\xi,\xi_0\right)\right|-r^2\leq 0.$$

When $\xi\in K'$, so

$$\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,w\left(z\right)+\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,f\left(z\right)\leq M+\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\left[-u\left(z\right)\right]=M-M=0$$

and finally, if $\xi\in K$, then

$$\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,w\left(z\right)+\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,f\left(z\right)\leq M+\overline{\lim_{\substack{z\to\xi\\ z\in L\cap G}}}\,\frac{\rho\left(z\right)}{\left|\rho_0\right|}M=M-M=0.$$

Thus, $w\left(z\right)+f\left(z\right)\leq 0$ in $L\cap G$ and since $w\in U_A\left(\varphi,G\right)$ n is arbitrary, then $\omega\left(z\right)+f\left(z\right)\leq 0$ on $L\cap G$. Hence it follows that

$$\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\,\omega\left(z\right)\leq-\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\,f\left(z\right)\leq-\left(-r^2-u\left(\xi_0\right)\right)=r^2+\frac{M\mu\left(K'\right)}{2\pi r}.$$

Choosing piece $K'=\left(\partial L\cap G\right)\backslash K$ so small that

$$\frac{M\mu\left(K'\right)}{2\pi r}<r^2.$$

Then

$$0\leq\overline{\lim_{\substack{z\to\xi\\ z\in G}}}\,\omega\left(z\right)\leq 2r^2$$

and for $r\to+0$ we get

$$\lim_{\substack{z\to\xi\\ z\in G}}\omega\left(z\right)=0.$$

The theorem is proved . □

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