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DIFFERENCE SCHEMES OF HIGH ACCURACY FOR EQUATION OF SPIN WAVES IN MAGNETS

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Abstract

Three-parameter difference schemes of the finite element method with a high order of accuracy are considered in the article for a mathematical model of spin waves in magnets (Sobolev-type equations). Discretization of time and space variables is conducted on the basis of the finite element method. The parameters of the scheme allow choosing the best approximation and accuracy, and an economic algorithm for numerical implementation. Theorems on the stability and convergence of the considered difference schemes are obtained.

Keywords: *Sobolev type equation, spin wave equation, difference schemes, finite element method, a priori estimates, stability, convergence, accuracy.*

Mathematics Subject Classification (2010): *35G05, 65N06.*

Introduction

The solution of complex applied problems requires the creation of more accurate numerical algorithms or the improvement of existing ones. This is especially manifested in the study of complex nonstationary processes, for example, in boundary value problems for linear partial differential equations, unsolved with respect to the highest time derivative, called Sobolev-type equations:

$$A_0 D_t^p u + \sum_{q=0}^{p-1} A_{p-q} D_t^q u = f$$

where $A_0, A_1, ..., A_p$—are the linear differential operators in spatial variables. The study of such equations began with the research conducted by S.L. Sobolev. These issues arise when solving problems in geophysics, oceanology, atmosphere physics, physics of magnetically ordered structures, plasma physics, physics of semiconductors, in problems related to wave propagation in media with strong dispersion, and in many other spheres [1]-[4].

Here are some examples.

1^0. Equation of two-temperature plasma in an external magnetic field: the equation describing low-frequency electron magneto-acoustic waves [1]

$$\frac{\partial^2}{\partial t^2}\Delta_2 u_e + \frac{q_{A_e}^2}{c^2}\frac{\partial^2}{\partial t^2}\left(\Delta_3 u_e - \frac{1}{r_{D_e}^2}u_e\right) + \omega_{B_e}^2\frac{\partial^2 u_e}{\partial x_3^2} = \frac{q_{A_e}^2}{c^2} f(x,t) \qquad (1)$$

or the equation describing low-frequency ionic magneto-acoustic waves

$$\frac{\partial^2}{\partial t^2}\Delta_2 u_i + \frac{q_{A_i}^2}{c^2}\frac{\partial^2}{\partial t^2}\left(\Delta_3 u_i - \frac{1}{r_{D_i}^2}u_i\right) + \omega_{B_i}^2\frac{\partial^2 u_i}{\partial x_3^2} = \frac{q_{A_i}^2}{c^2}f(x,t). \quad (2)$$

Here $u_e(x,t)-$ are the electric fields of electrons, $u_i(x,t)-$ are the electric fields of ions, $\Delta_2 = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}, \Delta_3 = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-$are two-dimensional and three-dimensional Laplace operators, $r_{D_e}^2 = T_e^2/\left(4\pi e^2 n_0\right)-$is the square of the electron Debye radius, $r_{D_i}^2 = T_i^2/\left(4\pi e^2 n_0\right)-$is the square of the ionic Debye radius, $q_{A_e} = B_0/\left(4\pi n_0 M\right)-$ is the Alfvén speed of electrons, $q_{A_i} = B_0/\left(4\pi n_0 M\right)-$ is the Alfvén speed of ions, $\omega_{B_e} = eB_0/\left(mc\right), \omega_{B_i} = ZeB_0/\left(Mc\right)-$ are the Larmor frequencies of electrons and ions (m and M are their masses, respectively), $c-$ is the speed of light in vacuum, $Z-$ is the ratio of the charges of an ion and an electron, B_0- is an external constant magnetic field, n_0- is the unperturbed density of particles, $e-$ is the absolute value of the charge of an electron, T_e- is the temperature of electrons, $T_i-$is the temperature of ions.

2^0. The equation of ion-acoustic waves in non-magnetized plasma [1]:

$$\frac{\partial^2}{\partial t^2}\left(\Delta_3 u - u\right) + \Delta_3 u = f(x,t). \quad (3)$$

3^0. The equation of spin waves in magnets [3]:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_1^2\right)\Delta_3 u(x,t) + \omega_2^2\Delta_2 u(x,t) = f(x,t), \quad (4)$$

where $\omega_1 = \gamma\left(H_0 + \beta M_0\right), \omega_2 = \gamma\sqrt{4\pi M_0\left(H_0 + \beta M_0\right)}, \gamma = g|e|/\left(2mc\right), g-$ is the hydromagnetic ratio of a ferromagnet, $\beta = K/M_0^2, M_0 = m_0 e_3-$ is a ferromagnet of the "easy" axis (e_3) type, $K-const., H_0 = H_0 e_3-$ is an external field.

4^0.The equation of spin waves in magnets of the “easy” plane type [3]:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_3^2\right)\Delta_3 u(x,t) + \omega_4^2 u_{x_2x_2}(x,t) + \omega_5^2 u_{x_3x_3}(x,t) = f(x,t). \quad (5)$$

where

$$\omega_3 = \gamma\sqrt{H_0\left(H_0 + |\beta|M_0\right)}, \quad \omega_4 = \gamma\sqrt{4\pi\left(H_0 + |\beta|M_0\right)M_0}, \quad \omega_5 = \gamma\sqrt{4\pi H_0 M_0},$$

$$u_{x_\alpha x_\alpha} = \frac{\partial^2 u}{\partial x_\alpha^2}, \alpha = 2,3.$$

Initial and some boundary conditions are added to equations $1^0 - 4^0$ The existence and uniqueness of the solution of the initial-boundary value problems for the above equations are considered in [1]-[4].

In [3, 5] similar problems with some transformation are reduced to two equations (one contains differentials in time, the other contains differentials in space); then these equations are solved by the finite difference method using quasi-uniform grids. The schemes constructed have the second-order of accuracy in time and in spatial variables with sufficient smoothness of the solution to the original differential problem.

In [6] initial-boundary value problems for equation (1), are studied, where difference schemes based on the finite element method are considered. Accuracy estimates $O\left(\tau^3+h^k\right)$ are obtained, where k is a polynomial degree of piecewise polynomial functions.

Similar studies are given in [7] for the equation of ion-acoustic waves in non-magnetized plasma. Accuracy estimates are also obtained there $O\left(\tau^3+h^k\right)$.

In this article, the authors consider the initial-boundary value problem for equation (4). Three-parameter difference schemes of high accuracy are constructed and investigated on the basis of the finite element method with piecewise cubic interpolation. The scheme parameters made it possible to obtain schemes of the fourth-order of accuracy in time and to implement economic algorithm. On the basis of a special technique, a priori estimates were obtained and theorems on the stability and convergence of the numerical algorithms under consideration were proved.

1 Statement of the problem

Equations (4) are written in the following form

$$\frac{\partial^2}{\partial t^2}\Delta_3 u+\omega_1^2\Delta_3 u+\omega_2^2\Delta_2 u=f(x,t),(x,t)\in Q_T \tag{6}$$

with initial conditions

$$u(x,t)=u_0(x,t),\frac{\partial u}{\partial t}(x,t)=u_1(x,t),t=0,x\in\overline{\Omega} \tag{7}$$

with boundary conditions

$$u(x,t)=\mu(x,t),x\in\Gamma=\partial\overline{\Omega},t\in(0,T], \tag{8}$$

where $\overline{\Omega}=\Omega+\Gamma$, $\quad \Omega=\{0<x_k<l_k,k=1,2,3\}$, $\quad Q_T=\{(x,t):x\in\Omega,t\in(0,T]\}$.

Let us formulate a generalized statement of the problem (6)-(8). We call function $u(x,t)$, a generalized solution of the problem, which for each $t\in(0,T)$ belongs to $H=\overset{\circ}{W}{}_2^1(\Omega)$ has derivative $\frac{\partial^2 u}{\partial t^2}\in W_2^1(\Omega)$ and satisfies almost everywhere on $(0,T)$ the following relations [8]:

$$a_3\left(\frac{d^2u(t)}{dt^2},\vartheta\right)+a_2\left(u(t),\vartheta\right)+a_1\left(u(t),\vartheta\right)=\left(f(t),\vartheta\right), \tag{9}$$

$$\left(u(0)-u_0,\vartheta\right)=0,\left(\frac{du}{dt}(0)-u_1,\vartheta\right)=0,\forall\vartheta(t)\in H, \tag{10}$$

where

$$a_3(u,\vartheta)=\int_\Omega\sum_{k=1}^{3}u_{x_k}\vartheta_{x_k}dx,a_2(u,\vartheta)=\omega_1^2a_3(u,\vartheta),a_1(u,\vartheta)=\omega_2^2\int_\Omega\sum_{k=1}^{2}u_{x_k}\vartheta_{x_k}dx.$$

Here $u = u(t)-$ is the function of the abstract argument $t \in [0, T]$ co with values in H, $\overset{\circ}{W}{}_2^1(\Omega)-$ is the Sobolev space with scalar product

$$(u(x), \vartheta(x)) = \int_\Omega \sum_{k=1}^{3} \frac{\partial u}{\partial x_k} \cdot \frac{\partial u}{\partial x_k} dx,$$

with norm

$$\|u\|_1 = \sqrt{\int_\Omega \sum_{k=1}^{3} (u_{x_k})^2\, dx}$$

and vanishing at boundary Γ of region $\overline{\Omega}$, $dx = dx_1 dx_2 dx_3$. It is obvious that $0 \le a_1(u, u) \le C_1\|u\|_1^2, c_2\|u\|_1^2 \le a_2(u, u) \le C_2\|u\|_1^2, c_3\|u\|_1^2 \le a_3(u, u) \le C_3\|u\|_1^2$, where C_1, C_2, C_3, c_2, c_3- are the positive constants.

2 Discretization in space and time

Let us approximate problem (9), (10) in spatial variables using the finite element method. Let $H_h \subset H$ be the set of elements of $\vartheta_h = \sum_{m=1}^{M} a_m \varphi_m(x)$ form. Here $\{\varphi_m = \varphi_m(x)\}_{m=1}^{M}$ is the basis of piecewise polynomial functions that are a degree p polynomial at each finite element [8, 9].

Let us give an example of a basis based on third-degree polynomials. Let us introduce a partition of domain Ω into $M = N_1 \times N_2 \times N_3$ parallelepipeds:

$$\Omega_{ijk} = \{(i-1)h_1 \le x_i \le ih_1, (j-1)h_2 \le x_2 \le jh_2, (k-1)h_3 \le x_3 \le kh_3\},$$

$$i = \overline{1, N_1}, j = \overline{1, N_2}, k = \overline{1, N_3}, h_3 = l_s/N_s, s = 1, 2, 3.$$

Let us choose a system of basis functions:

$$\Phi_{ijk}(x_1, x_2, x_3) = \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3), i = \overline{1, N_1}, j = \overline{1, N_2}, k = \overline{1, N_3},$$

where $\varphi_l(x)-$ is the basis function based on the B_3- spline [8]. In this case $p = 3$. The approximate solution can be represented as a bicubic spline:

$$u_h(x_1, x_2, x_3, t) = \sum_{k=1}^{N} a_k(t)\Phi_k(x_1, x_2, x_3). \tag{11}$$

Therefore, in accordance with (9), (10) we obtain a semi-discrete problem at $t \in [0, T]$:

$$a_3\left(\frac{d^2 u_h(t)}{dt^2}, \vartheta_h\right) + a_2(u_h, \vartheta_h) + a_1(u_h, \vartheta_h) = (f(t), \vartheta_h), \tag{12}$$

$$(u_h(0)-u_0,\vartheta_h)=0,\left(\frac{du_h}{dt}(0)-u_1,\vartheta_h\right)=0,\forall\vartheta_h\in H_h. \tag{13}$$

Problem (12), (13) corresponds to the Cauchy problem:

$$D\frac{d^2u_h(t)}{dt^2}+Au_h(t)=f_h(t),u_h(0)=u_{0,h},\frac{du_h}{dt}(0)=u_{1,h}. \tag{14}$$

Operators D,A act from H_h in H_h. They correspond to stiffness matrices $D=(a_3(\varphi_l,\varphi_m))_{l,m=1}^M$ and $A=(a_2(\varphi_l,\varphi_m))_{l,m=1}^M+(a_1(\varphi_l,\varphi_m))_{l,m=1}^M$. In addition, $u_{k,h}=P_hu_k(x),k=0,1$, where P_h- is the projection operator $P_hH=H_h$. Here

$$D=D^*>0,A=A^*>0. \tag{15}$$

We approximate problem (14) in time by the difference scheme [10]:

$$\begin{cases} (D-\gamma\tau^2A)\dfrac{\hat{\dot{y}}-\dot{y}}{\tau}+A\dfrac{\hat{y}+y}{2}=\varphi_1, \\ (D-\alpha\tau^2A)\dfrac{\hat{y}-y}{\tau}-\left(D-\beta\tau^2A\right)\dfrac{\hat{\dot{y}}+\dot{y}}{2}=\varphi_2, \end{cases} \tag{16}$$

$$y^0=u_0,\dot{y}^0=u_1.$$

Here $y=y^n=y(t_n)$, $\hat{y}=y^{n+1}$, $\dot{y}=\dot{y}^n=\dfrac{dy}{dt}(t_n),n=0,1,\dots,\ y^n,\dot{y}^n\in H_h$, $\varphi_k=\displaystyle\int_0^1 f(t_n+\tau\xi)\vartheta_k(\xi)d\xi$, $k=1,2$, $\vartheta_1(\xi)=1$, $\vartheta_2(\xi)=s_1\vartheta_2^{(1)}(\xi)+s_2\vartheta_2^{(2)}(\xi)$, $\xi=\dfrac{t-t_n}{\tau}$, $\vartheta_2^{(1)}(\xi)=\tau\left(\xi-\dfrac{1}{2}\right)$, $\vartheta_2^{(2)}(\xi)=\tau\left(\xi^3-\dfrac{3}{2}\xi^2+\dfrac{1}{2}\xi\right)$, $s_1=180\beta-40\alpha$, $s_2=1680\beta-280\alpha$.

Parameters α,β,γ obey the condition of the fourth-order approximation

$$\alpha+\gamma=\beta+1/6. \tag{17}$$

In [10], based on the analysis of variance, it was proved that under additional conditions

$$\beta-6\alpha\gamma+1/40=0, \tag{18}$$

scheme (16) has an order of accuracy $O(\tau^6)$.

In addition, according to the values calculated

$$y^n=y(t_n),\dot{y}^n=\frac{dy}{dt}(t_n),y^{n+1}=y(t_{n+1}),\dot{y}^{n+1}=\frac{dy}{dt}(t_{n+1})$$

it is possible to restore the approximation for $u(t)$ at any point in time $t\in[t_n,t_{n+1}]$, $n=0,1,\dots$ by the following formula

$$y(t)=y^n\varphi_{00}^n(t)+\dot{y}^n\varphi_{10}^n(t)+y^{n+1}\varphi_{01}^n(t)+\dot{y}^{n+1}\varphi_{11}^n(t),$$

where

$$\varphi_{00}^n(t) = 2\xi^3 - 3\xi^2 + 1, \varphi_{01}^n(t) = 3\xi^2 - 2\xi^3,$$

$$\varphi_{10}^n(t) = \tau(\xi^3 - 2\xi^2 + \xi), \varphi_{11}^n(t) = \tau(\xi^3 - \xi^2), \xi = (t - t_n)/\tau.$$

In this case, the errors of the scheme $\|y(t) - u(t)\|$ and $\|\dot{y}(t) - \dot{u}(t)\|$ at any point in time $t \in [t_n, t_{n+1}], n = 0, 1, \ldots$ remain the same.

3 Study of stability and convergence

Let us analyze the stability and accuracy of the scheme (16). The following theorem holds.

Theorem 1. *[10] Let $A^* = A > 0, D^* = D > 0$ and the approximation conditions* (17) *are satisfied. If*

$$D - \delta\tau^2 A \geq \varepsilon D, 0 < \varepsilon < 1, \delta = max\{\alpha, \beta, \gamma\}, \tag{19}$$

then the solution $y(t)$ of scheme (16) *converges to the solution of problem* (14) *$u_h(t) \in C^6[0, T]$ and the following estimate is true:*

$$\|u_h(t) - y(t)\|_A + \|\dot{u}_h(t) - \dot{y}(t)\|_D \leq M\tau^4.$$

The proof is based on reducing the two-layer vector scheme (16) to a three-layer scheme, separately for the solution of y and its derivative $\dot{y}$. The proof implies the permutation of operators A and D, i.e. $AD = DA$. To eliminate this condition, instead of $y, \dot{y}$ we introduce $w = D^{1/2}y, \dot{w} = D^{1/2}\dot{y}$, where $D^{1/2}-$ is the square root of positive operator D. Note that $(D^{1/2})^* = D^{1/2} > 0$ and there is an inverse operator $D^{-1/2} = (D^{-1/2})^* > 0$. After obvious transformations, we obtain the following scheme from (16)

$$\begin{cases} \widetilde{D}_\gamma \dfrac{\hat{\dot{w}} - \dot{w}}{\tau} + \widetilde{A}\dfrac{\hat{w} + w}{2} = \widetilde{\varphi}_1, \\ \widetilde{D}_\alpha \dfrac{\hat{w} - w}{\tau} - \widetilde{D}_\beta \dfrac{\hat{\dot{w}} + \dot{w}}{2} = \widetilde{\varphi}_2, \end{cases} \tag{20}$$

where $\widetilde{\varphi}_1 = D^{-1/2}\varphi_1, \widetilde{\varphi}_2 = D^{-1/2}\varphi_2, \widetilde{D}_\omega = \widetilde{D} - \omega\tau^2\widetilde{A}, \widetilde{D} = E, \widetilde{A} = D^{-1/2}AD^{-1/2}$, $\omega = \alpha, \beta, \gamma$. It is clear that $\widetilde{D} = \widetilde{D}^* > 0$, $\widetilde{A} = \widetilde{A}^* > 0$ and $\widetilde{D}\widetilde{A} = \widetilde{A}\widetilde{D}$.

To estimate the accuracy of scheme (20), it is necessary to estimate the error $z = w - u_h$. Using the technique of such an estimation in the theory of difference schemes [11] and the theory of the finite element method [12], we formulate the result.

Theorem 2. *Under condition* (17), (19) *the solution of scheme* (20) *$w(t)$ converges to the solution of problem* (6)-(8) *and the following estimate is true:*

$$\|u(t) - y(t)\|_1 \leq M(h^\sigma + \tau^4).$$

When choosing a degree $\sigma = 3$ polynomial on each finite element in space, we obtain the third-order of accuracy in step h.

Let us check the fulfillment of the stability condition (19). We represent the operators of scheme (16) in the following form

$$D = A_1 + A_2 + A_3, A = \omega_1^2 A_1 + \omega_1^2 A_2 + \omega_1^2 A_3 + \omega_2^2 A_1 + \omega_2^2 A_2,$$

where operators $A_k \geq 0$ correspond to stiffness matrices $A_k = (b_k(\varphi_l, \varphi_m))_{l,m=1}^M$ with bilinear form $b_k(u, \vartheta) = \int_\Omega (u_{x_k}, \vartheta_{x_k})\, dx$. Condition (19) takes the following form

$$(1-\varepsilon)(A_1 + A_2 + A_3) - \delta\tau^2 \left(\omega_1^2 A_1 + \omega_1^2 A_2 + \omega_1^2 A_3 + \omega_2^2 A_1 + \omega_2^2 A_2\right) \geq 0.$$

To satisfy it, it is enough that

$$\tau^2 \leq \frac{1-\epsilon}{\delta} max\left(\frac{1}{\omega_1^2}, \frac{1}{\omega_2^2}\right), 0 < \varepsilon < 1. \tag{21}$$

The last condition is interesting because the step in time is not related to the step in space and is determined by the following parameters of the problem: ω_1^2 and ω_2^2- are the Väisälä-Brent frequencies. For the parameters of the scheme (16) $\alpha = 1/8, \beta = 1/24, \gamma = 1/12$ we have $\delta = 1/8$. So, finally $\tau \leq \frac{2\sqrt{2(1-\varepsilon)}}{min(\omega_1, \omega_2)}$.

4 Algorithm for implementation

When solving a specific problem, we use the scheme (16). The algorithm for implementing the scheme is reduced to solving two equations on each layer:

$$Cy^{n+1} = F_1, C\dot{y}^{n+1} = F_2,$$

where $C = D^2 - \left(\alpha - \frac{1}{6}\right)\tau^2 AD + \left(\alpha\gamma - \frac{\beta}{4}\right)\tau^4 A^2$, and the right-hand sides are calculated from the known values of $y^n, \dot{y}^n$ and φ_1, φ_2.

The matrix C is factorized: $C = C_1 C_2 = (D - \lambda_1 A)(D - \lambda_2 A)$, where λ_1, λ_2 are the roots of the equation

$$\lambda^2 + \left(\alpha + \gamma - \frac{1}{4}\right)\lambda + \left(\alpha\gamma - \frac{\beta}{4}\right) = 0.$$

For $\alpha = \frac{1}{8}$, $\beta = \frac{1}{24}$, $\gamma = \frac{1}{12}$, we have $\lambda_1 = -\frac{1}{24}, \lambda_2 = 0$, i.e. when condition $\alpha\gamma - \frac{1}{4} = 0$ is satisfied, the algorithm of scheme (16) becomes an economic one. Then $y^{n+1} = C^{-1}F_1 = C_2^{-1}C_1^{-1}F_1, \dot{y}^{n+1} = C_2^{-1}C_1^{-1}F_2$. For the inversion of matrix C_1, C_2 for example, the direct square root method is applied once, at the initial point in time. On the remaining layers, the solution is found by multiplying the matrix $C^{-1} = C_2^{-1}C_1^{-1}$ by vectors F_1, F_2.

The main number of operations for scheme (16) falls on the inversion of one operator D on the first layer and the subsequent multiplication of vector $f_h^n - Ay^n$ by D^{-1} on the remaining layers.

For scheme (16) we invert two operators $C_1 = D - \omega_1 A$ and $C_2 = D - \omega_2 A$. on the first layer. On the remaining layers, we multiply two vectors F_1, F_2 by $C^{-1} = C_2^{-1}C_1^{-1}$. Thus, the number of arithmetic operations for the implementation of scheme (16) is approximately 4 times greater than the classical three-layer schemes of the second-order of accuracy. This scheme allows choosing large time steps to achieve a certain accuracy since there is no limitation on time steps.

The increase in computing time for scheme (16) occurs due to the need to construct the stiffness matrices D and A, realized by numerical integration (one-time at initial time $t = 0$).

5 Conclusions

A fourth-order of accuracy method for solving the problem for the equation of spin waves in magnets was developed and investigated. It was based on finite element approximation in space and time using third degree polynomials. An algorithm for the implementation of the method was developed. Theorems on stability and convergence of the considered difference schemes were proved. A separate study will be devoted to numerical modeling, where, on the basis of the algorithm implementation of the method developed here, it will be tested for exact solutions in the form of a Fourier series and compared with other methods. In addition, on the basis of a computational experiment, the convergence rates of the method in time-space directions will be checked, and the visualization will be done, which will confirm these theoretical results.

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