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WEIGHTED (m, δ)–GREEN FUNCTIONS IN $\mathbb{C}^n$

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Abstract

In this work some extremal function and its properties are studied for the class of m-subharmonic functions. We study weighted (m, δ)-Green function $V_m^*(z, K, \psi, \delta)$, defined by the class

$$\mathcal{L}_m^\delta = \{u(z) \in sh_m(\mathbb{C}^n) : \ u(z) \le \delta, \ z \in \mathbb{C}^n\}, \ \delta > 0.$$

We see that the regularity of the points with respect to different numbers δ differ from each other. Nevertheless, we will prove that if the compact $K \subset \mathbb{C}^n$ is (m, δ, ψ)-regular, then weighted (m, δ)-Green function is continuous in the whole space $\mathbb{C}^n$.

Keywords: *m-Green function, weighted m-Green function, weighted (m, δ)-Green function, m-regular compact.*

Mathematics Subject Classification (2010): *31B05, 31B15 , 31C10, 31C05.*

Introduction

For a twice continuously differentiable function $u \in C^2(D)$, where $D \subset \mathbb{C}^n$ and $m = 1, 2, \dots, n$ the elementary complex Hessian operator is defined by

$$H_m(u) = \sum_{1 \le j_1 < \dots < j_m \le n} \lambda_{j_1} \dots \lambda_{j_m},$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the complex Hessian $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$. Using the operators $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2}(\partial - \overline{\partial})$, so that $dd^c = \frac{i}{2}\partial\overline{\partial}$, one gets

$$(dd^c u)^m \wedge \beta^{n-m} = \ m!\,(n-m)! H_m(u)\,\beta^n$$

where $\beta = dd^c|z|^2$ standard volume form in $\mathbb{C}^n$. Consider a set

$$\Gamma_m = \{H_1(\lambda) \ge 0, ..., H_m(\lambda) \ge 0\}.$$

Let $\alpha = \frac{i}{2} \sum\limits_{j,k} a_{j\bar{k}} dz_j \wedge d\bar{z}_k$ be a real differential form of bidegree $(1, 1)$, where is $\left(a_{j\bar{k}}\right)$ is a Hermitian matrix. If $\lambda(\alpha) = (\lambda_1(\alpha), ..., \lambda_n(\alpha))$ are the eigenvalues of this matrix, then the Hessian $H_m(\alpha) = H_m(\lambda(\alpha))$ is defined. This relation between Hermitian $(1, 1)$-forms and vectors $\lambda \in \mathbb{R}^n$ allows us to define

$$\overset{\wedge}{\Gamma}_m = \{\alpha : \ \lambda(\alpha) \in \Gamma_m\} = \left\{H_m\left(\alpha + t_1 dd^c|z_1|^2 + ... + t_n dd^c|z_n|^2\right) \ge 0 \ \forall t = (t_1, ..., t_n) \in \mathbb{R}_+^n\right\} =$$
$$= \{\alpha \wedge \beta^{n-1} \ge 0, ..., \alpha^m \wedge \beta^{n-m} \ge 0\}.$$

Definition 0.1. *A twice continuously differentiable function $u \in C^2(D)$, where $D \subset \mathbb{C}^n$, is said to be m-subharmonic (briefly, m-sh) at a point $z^0 \in D$ if the eigenvalues $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\Big|_{z=z_0}$ belong to Γ_m or, which is the same, $dd^c u \in \overset{\wedge}{\Gamma}_m$ $(1 \le m \le n)$. A function $u \in C^2(D)$ is called an m-sh function in D if it is an m-sh function at every point $z^0 \in D$.*

In other words, a function $u \in C^2(D)$ is called an m-sh function if $dd^c u \in \overset{\wedge}{\Gamma}_m$, which is equivalent to the condition

$$(dd^c u)^k \wedge \beta^{n-k} \ge 0 \;\forall\; k = 1, 2, ..., m.$$

Definition 0.2. *A function $u \in L^1_{loc}(D)$ is said to be m-subharmonic (m-sh) in a domain $D \subset \mathbb{C}^n$ if it is upper semicontinuous and, for any twice continuously differentiable m-sh functions v_1 , ... , v_{m-1} the current $dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}$ defined as*

$$\left[dd^c u \wedge dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m}\right](\omega) = \int_{\omega \in F^{0,0}} u\, dd^c v_1 \wedge ... \wedge dd^c v_{m-1} \wedge \beta^{n-m} \wedge dd^c \omega$$

is positive.

The set of $m - sh$ functions in D is denoted by $sh_m(D)$. Note that for $m = 1$ the class sh_m coincides with the class of subharmonic (sh) functions, and for $m = n$ it coincides with the class of plurisubharmonic (psh) functions. See details [1, 2, 3, 4].

1 Weighted m-Green function in $\mathbb{C}^n$

Let $\psi(z)$ be a bounded function on the compact set $K \subset \mathbb{C}^n$, $\psi(z) < 1$. Consider the class of functions

$$\mathcal{L}_m(K, \psi) := \{u(z) \in \mathcal{L}_m,\; u(z)|_K \le \psi(z)\},$$

where $\mathcal{L}_m = \{u \in sh_m(\mathbb{C}^n) : u \le 1\}$. If $m = n$, then $sh_m = psh$ and the class $\mathcal{L}_m$ is trivial, $\mathcal{L}_m = \{const\}$. However, in the class of the psh functions the weighted psh-Green functions we studied in our previous articles (see [10, 11]), using the Lelong class. Therefore, we can assume $1 \le m < n$. Then $\mathcal{L}_m$ is wide class and we put

$$V_m(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}_m(K, \psi)\},\; z \in \mathbb{C}^n.$$

The regularization $V^*_m(z, K, \psi) = \overline{\lim_{w \to z}} V_m(w, K, \psi)$ is said to be the weighted m-Green function of K with respect to $\psi(z)$.

Note that in the case $\psi(z) \equiv 0$, the function $V^*_m(z, K, \psi)$ coincides with the m-Green function $V^*_m(z, K)$, i.e., $V^*_m(z, K, 0) \equiv V^*_m(z, K)$. The Green functions $V^*_m(z, K)$ where studed in the article of Abdullaev, Sharipov [13].

It is not difficult to prove the following simple properties of the weighted m-Green function:

1°. $V_m^*(z,K,\psi) \in \mathcal{L}_m$

2°. If $K_1 \subset K_2$, then $V_m^*(z,K_1,\psi) \geq V_m^*(z,K_2,\psi)$.

3°. If $\psi_1 \leq \psi_2 \ \forall z \in K$, then $V_m^*(z,K,\psi_1) \leq V_m^*(z,K,\psi_2)$.

Lemma 1.1. *For any compact set K, the following inequalities hold:*

$$\inf_{z\in K}\psi(z)+(1-\inf_{z\in K}\psi(z))\cdot V_m^*(z,K) \leq V_m^*(z,K,\psi) \leq \sup_{z\in K}\psi(z)+(1-\sup_{z\in K}\psi(z))\cdot V_m^*(z,K). \tag{1}$$

Proof. Indeed, if $u \in \mathcal{L}_m(K,\psi)$ i.e., $u \in \mathcal{L}_m$, $u|_K \leq \psi$, then $\dfrac{u(z)-\sup_{z\in K}\psi(z)}{1-\sup_{z\in K}\psi(z)} \in \mathcal{L}_m(K)$. Here we assume that $\sup_{z\in K}\psi(z) < 1$. Consequently,

$$\frac{u(z)-\sup_{z\in K}\psi(z)}{1-\sup_{z\in K}\psi(z)} \leq V_m^*(z,K) \quad \forall z \in \mathbb{C}^n$$

and

$$\frac{V_m^*(z,K,\psi)-\sup_{z\in K}\psi(z)}{1-\sup_{z\in K}\psi(z)} \leq V_m^*(z,K) \quad \forall z \in \mathbb{C}^n.$$

From here we obtain

$$V_m^*(z,K,\psi) \leq (1-\sup_{z\in K}\psi(z))\cdot V_m^*(z,K)+\sup_{z\in K}\psi(z) \quad \forall z \in \mathbb{C}^n.$$

On the other side, if $u \in \mathcal{L}_m(K)$, then $(1-\inf_{z\in K}\psi(z))u(z)+\inf_{z\in K}\psi(z) \in \mathcal{L}_m(K,\psi)$. Hence,

$$\inf_{z\in K}\psi(z)+(1-\inf_{z\in K}\psi(z))\cdot V_m^*(z,K) \leq V_m^*(z,K,\psi) \quad \forall z \in \mathbb{C}^n,$$

so (1) is hold. □

Corollary 1.1. $V_m^*(z,K,\psi) \equiv 1$ *if and only if* $V_m^*(z,K) \equiv 1$.

From the fact, that $V_m^*(z,K) \equiv 1$ if and only if K is an m-polar set [13], it follows $V_m^*(z,K,\psi) \equiv 1$ if and only if K is m-polar set.

If a function $\psi(z)$ extends to the space $\mathbb{C}^n$ as a function from the class $\mathcal{L}_m$ i.e. if there is a function

$$\Psi \in \mathcal{L}_m : \Psi|_K \equiv \psi, \tag{2}$$

then it is obvious $V_m(z,K,\psi) \geq \Psi(z)$ and

$$V_m(z,K,\psi) \equiv \psi(z), \ z \in K \tag{3}$$

Below we assume that the m-Green function satisfies the condition (3).

Definition 1.1. *We say that a compact K is globally (m,ψ)-regular at z^0 if $V_m^*(z^0,K,\psi) = \psi(z^0)$. We say that a compact K is locally (m,ψ)-regular at z^0 if $V_m^*(z^0,K\cap\overline{B}(z^0,r),\psi) = \psi(z^0)$ for every ball $B(z^0,r), r>0$.*

Theorem 1.1. *Let $\psi(z)$ is continuous function on K. If K is globally (m,ψ)-regular i.e. if K is globally (m,ψ)-regular at arbitrary point $z^0\in K$, then $V_m^*(z,K,\psi) = V_m(z,K,\psi)$ and $V_m^*(z,K,\psi)$ continuous in $\mathbb{C}^n$.*

Proof. Since $\psi(z)$ is a continuous function on K, $\psi(z)$ can be extended continuously throughout K, i.e., there is a function $\Psi(z)\in C(\mathbb{C}^n)$ such that $\Psi(z)|_K = \psi(z)$ (see Whitney H. [12]). We use the standard approximation $u_j\downarrow V_m^*(z,K,\psi)$, where $u_j\in\mathcal{L}_m\cap C^\infty(\mathbb{C}^n)$. Since $V_m^*(z,K,\psi)\equiv\Psi(z),\ z\in K$, then for any $\varepsilon>0$ there is an open set $\{z\in\mathbb{C}^n,\ V_m^*(z,K,\psi)<\Psi(z)+\varepsilon\}$ contained K. Therefore, by the Hartogs lemma, there exists $j_0\in\mathbb{N}$ such that

$$u_j(z)<\Psi(z)+2\varepsilon=\psi(z)+2\varepsilon,\ \forall z\in K,\ j>j_0.$$

From here, $u_j-2\varepsilon\in\mathcal{L}_m(\psi,K)$ and

$$u_j-2\varepsilon\le V_m(z,K,\psi)\le V_m^*(z,K,\psi)\le u_j,\ j>j_0,\ z\in\mathbb{C}^n.$$

This means that the sequence u_j converges to $V_m^*(z,K,\psi)$ uniformly and $V_m^*(z,K,\psi) = V_m(z,K,\psi)\in C(\mathbb{C}^n)$. □

Theorem 1.2. *Let K be a compact set, and $\psi(z)$ is weight on $K:\psi(z)\in C(K)$. Then K is locally (m,ψ)-regular at $z^0\in K$ if and only if K is locally m-regular (case $\psi\equiv 0$) at z^0.*

Proof. Indeed, we use the inequality (1). If the point $z^0\in K$ is not locally m-regular, i.e., if $V_m^*(z^0,K\cap\overline{B})=\sigma>0$ for some neighborhood $B:\ \ z^0\in B\subset\mathbb{C}^n$, then $V_m^*(z^0,K\cap\overline{B}_1)\ge\sigma$ for any $z^0\in B_1\subset B$. Therefore, by (1)

$$V_m^*(z^0,K\cap\overline{B}_1,\psi(z^0))\ge(1-\inf_{z\in K\cap\overline{B}_1}\psi(z))\cdot V_m^*(z^0,K\cap\overline{B}_1)+\inf_{z\in K\cap\bar{B}_1}\psi(z)$$

and

$$V_m^*(z^0,K\cap\overline{B}_1,\psi(z^0))\ge\sigma\cdot\left(1-\inf_{z\in K\cap\overline{B}_1}\psi(z)\right)+\inf_{z\in K\cap\overline{B}_1}\psi(z) \tag{4}$$

Since $\psi(z)$ is continuous, then choosing the neighborhood B_1 small enough we can make the right part of (4) to be greater, than $\psi(z^0)$ i.e., $V_m^*(z,K\cap B_1,\psi)>\psi(z^0)$ and the point z^0 is not locally (m,ψ)-regular.

Reversing the roles of $V_m^*(z,K\cap\overline{B}_1,\psi)$ and $V_m^*(z,K\cap\overline{B}_1)$ from (1) we can prove the second part of the theorem: if the point $z^0\in K$ is not locally (m,ψ)-regular, then it is not locally m-regular. □

2 Weighted (m,δ)-Green function

We fix the number $\delta > 0$ and define an (m,δ)-Green function. Let K be a compact set and ψ some bounded function on K. We assume $\psi(z) \le \delta$. Consider the class

$$\mathcal{L}_m^\delta := \{u(z) \in sh_m(\mathbb{C}^n) : \ u(z) \le \delta, \ z \in \mathbb{C}^n\}.$$

It is clear that if $u(z) \in \mathcal{L}_m$, then $c \cdot u(z) \in \mathcal{L}_m^\delta \ \forall c \in (0,\delta]$. We put

$$\mathcal{L}_m^\delta(K,\psi) := \{u(z) \in \mathcal{L}_m^\delta, \ u(z)|_K \le \psi(z)\}.$$

Definition 2.1. *The function $V_m^*(z,K,\psi,\delta) = \overline{\lim_{w\to z}} V_m(w,K,\psi,\delta)$ is called the (m,δ)-Green function of the compact set K with respect to ψ, where*

$$V_m(z,K,\psi,\delta) := \sup\{u(z) : u(z) \in \mathcal{L}_m^\delta(K,\psi)\}, \ z \in \mathbb{C}^n.$$

Note that, for $\delta = 1$ the (m,δ)-Green function coincides with the weighted m-Green function of the compact set K i.e., $V_m^*(z,K,\psi,1) \equiv V_m^*(z,K,\psi)$, for $\delta = 1, \psi(z) = 0$ coincides with the m-Green function, i.e. $V_m^*(z,K,0,1) \equiv V_m^*(z,K)$.

Let us list the properties of the (m,δ)-Green function:

1°. If $\delta_1 \le \delta_2$, then $V_m^*(z,K,\psi,\delta_1) \le V_m^*(z,K,\psi,\delta_2)$.

2°. If $\psi_1 \le \psi_2 \ \forall z \in K$, then $V_m^*(z,K,\psi_1,\delta) \le V_m^*(z,K,\psi_2,\delta)$.

3°. $V_m^*(z,K,\psi,\delta) = \delta \cdot V_m^*\left(z,K,\dfrac{\psi}{\delta}\right)$ in particular, $V_m^*(z,K,\delta) = \delta \cdot V_m^*(z,K)$.

Lemma 2.1. *For any compact set K, the following inequalities hold:*

$$\inf_{z\in K}\psi(z)+(\delta-\inf_{z\in K}\psi(z))\cdot V_m^*(z,K) \le V_m^*(z,K,\psi,\delta) \le \sup_{z\in K}\psi(z)+(\delta-\sup_{z\in K}\psi(z))\cdot V_m^*(z,K). \tag{5}$$

Proof. Really, if $u \in \mathcal{L}_m^\delta(K,\psi,\delta)$ i.e., $u \in \mathcal{L}_m^\delta, \quad u|_K \le \psi$, then $\dfrac{u(z) - \sup_{z\in K}\psi(z)}{\delta - \sup_{z\in K}\psi(z)} \in \mathcal{L}_m^\delta(K)$. Here we assume that $\sup_{z\in K}\psi(z) < \delta$. If $\sup_{z\in K}\psi(z) = \delta$, then the right side of the inequality (5) is also true. Consequently,

$$\frac{u(z) - \sup_{z\in K}\psi(z)}{\delta - \sup_{z\in K}\psi(z)} \le V_m^*(z,K) \quad \forall z \in \mathbb{C}^n$$

and

$$\frac{V_m^*(z,K,\psi,\delta) - \sup_{z\in K}\psi(z)}{\delta - \sup_{z\in K}\psi(z)} \le V_m^*(z,K) \quad \forall z \in \mathbb{C}^n.$$

From here

$$V_m^*(z,K,\psi,\delta) \le (\delta - \sup_{z\in K}\psi(z)) \cdot V_m^*(z,K) + \sup_{z\in K}\psi(z) \quad \forall z \in \mathbb{C}^n.$$

On the other side, if $u \in \mathcal{L}_m^\delta(K)$, then $(\delta - \inf_{z\in K}\psi(z))u(z) + \inf_{z\in K}\psi(z) \in \mathcal{L}_m^\delta(K,\psi)$. Hence,

$$\inf_{z\in K}\psi(z) + (\delta - \inf_{z\in K}\psi(z)) \cdot V_m^*(z,K) \le V_m^*(z,K,\psi,\delta) \quad \forall z \in \mathbb{C}^n.$$

The lemma is proved. □

Note that in the general case $V_m(z,K,\psi,\delta)$ and weight function ψ does not have to be exactly equal on K for all δ. In other words, condition (3) may not be satisfied.

Example 2.1. *Let $K = \overline{B(0,1)} \subset \mathbb{C}^2$ and $\psi(z) = |z|^2$. Then for $m = 1$ one can prove, that*

$$V_1(z,K,\psi,\delta) = \begin{cases} |z|^2, & |z| \le \sqrt{\frac{\delta}{2}}, \\ \delta - \dfrac{\delta^2}{4|z|^2}, & |z| > \sqrt{\frac{\delta}{2}}. \end{cases}$$

We see $V_1(z,K,\psi,\delta) = |z|^2,\ \forall z \in \left\{|z| \le \sqrt{\frac{\delta}{2}}\right\}$ and $V_m(z,K,\psi,\delta) < |z|^2,\ \forall z \in \left\{\sqrt{\frac{\delta}{2}} < |z| \le 1\right\}$.

We denote by $\Lambda = \Lambda(K,\psi)$ the set of numbers δ, for which the equality type (3) holds, i.e.

$$\Lambda = \Lambda(K,\psi) = \{\delta > 0 : V_m(z,K,\psi,\delta)|_K \equiv \psi(z)\}.$$

For Example 1, $\Lambda = [2,+\infty)$. In fact,

$$V_1(z,K,\psi,2) = \begin{cases} |z|^2, & |z| \le 1, \\ 2 - \dfrac{1}{|z|^2}, & |z| > 1. \end{cases}$$

So that $V_1(z,K,\psi,2)|_K \equiv \psi(z)$ and by property 1° of the section 2 $V_1(z,K,\psi,\delta) \ge V_1(z,K,\psi,2)$ for all $\delta \in [2,+\infty)$. If $\delta \in (0,2)$ then there is a point $z^0 \in K$ such that $V_1(z^0,K,\psi,\delta) < \psi(z^0)$, that is $(0,2) \cap \Lambda = \emptyset$.

The sets Λ may be empty.

Example 2.2. *Let $K = \{|z| \le 1\} \subset \mathbb{C}^2$ and $\psi(z) = 1 - |z|^2$. By the maximum principle and by the property 3° we have*

$$V_m(z,K,\psi,\delta) = V_m(z,K,\delta) = \delta V_m(z,K) = \delta \max\left\{0, \left(1 - \frac{1}{|z|^2}\right)\right\}.$$

Therefore, for any $\delta > 0$, $V_m(z,K,\psi,\delta) < \psi(z)$, $\forall |z| < 1$. That is in this case $\Lambda = \emptyset$.

If $\psi(z) \equiv c$, where c is constant, then $V_m(z,K,c,\delta) = c + V_m(z,K,\delta) = c + \delta V_m(z,K)$. Since, the m-Green function $V_m(z,K) \ge 0$, then for any $\delta > 0$ and $z \in K$ the inequalitie $V_m(z,K,c,\delta) \ge c$ holds. This means $\Lambda = (0,+\infty)$.

Let $\Lambda \neq \emptyset$. Then from the 1° property we easily get $\delta_1 \in \Lambda$, for $\delta_1 > \delta$, if $\delta \in \Lambda$.

Proposition 2.1. *If $\delta_j \in \Lambda, \ \forall j \in \mathbb{N}$ and $\delta_j \downarrow \delta_0 \neq 0$ as $j \to \infty$, then $\delta_0 \in \Lambda$.*

$\lhd$ Indeed, by the condition of the proposition, we have $V_m(z, K, \psi, \delta_j) = \psi(z), z \in K$. Using the properties 2° and 3°, we get

$$V_m(z, K, \psi, \delta_j) = \delta_j \cdot V_m \left(z, K, \frac{\psi}{\delta_j} \right) \leq \delta_j \cdot V_m \left(z, K, \frac{\psi}{\delta_0} \right).$$

Consequently, $\forall j \in \mathbb{N}$ we have

$$\psi(z) = V_m(z, K, \psi, \delta_j) \leq \delta_j \cdot V_m \left(z, K, \frac{\psi}{\delta_0} \right), \ z \in K.$$

Tending j to infinity, we get

$$\psi(z) \leq \delta_0 \cdot V_m \left(z, K, \frac{\psi}{\delta_0} \right) = V_m(z, K, \psi, \delta_0), \ z \in K,$$

i.e. $\psi(z) = \delta_0 \cdot V_m \left(z, K, \frac{\psi}{\delta_0} \right) = V_m(z, K, \psi, \delta_0), \ z \in K$ and $\delta_0 \in \Lambda$. $\rhd$

Proposition 2.1 follows, that if $\Lambda \neq \emptyset$ then both or $\Lambda = (0, \infty)$, or $\Lambda = [\delta_0, +\infty), \delta_0 > 0$. Note that if $\delta \in \Lambda(K, \psi)$, then $V_m(z, K, \psi, \delta) = \psi(z), \ z \in K$. Therefore, by the property of monotony $V_m(z, K \cap \overline{B}, \psi, \delta) = \psi(z), \ z \in K \cap \overline{B}$, for any ball $B \cap K \neq \emptyset$. It follows, that if $\delta \in \Lambda(K, \psi)$, then $\delta \in \Lambda(K \cap B, \psi)$.

3 (m, δ)-regularity

3.1 Unweighted case

We define (m, δ)-regularity and (m, δ)-local regularity in a similar way as for usual m-regularity.

Definition 3.1. *A compact set K is called globally (m, δ)-regular at a point $z^0 \in K$ if $V_m^*(z^0, K, \delta) = 0$. It is called locally (m, δ)-regular at a point $z^0 \in K$ if $V_m^*(z^0, K \cap \overline{B}(z^0, r), \delta) = 0$ for any $r > 0$.*

The proposition below immediately follows from property 3 and shows that (m, δ)-regularity is the same as classical m-regularity, namely, $\delta = 1$.

Proposition 3.1. *Let K be a compact subset of $\mathbb{C}^n$.*

(i) *A compact set K is globally (m, δ)-regular at a point $z^0 \in K$ if and only if it is globally m-regular at a point z^0.*

(ii) *A compact set K is locally (m, δ)-regular at a point $z^0 \in K$ if and only if it is locally m-regular at a point z^0.*

3.2 Weighted case

Definition 3.2. *Let $\delta \in \Lambda(\psi, K)$. The compact K is called globally (m, δ, ψ)-regular at point $z^0 \in K$ if $V_m^*(z^0, K, \psi, \delta) = \psi(z^0)$. It is called locally (m, δ, ψ)-regular at point $z^0 \in K$ if $V_m^*(z^0, K \cap \overline{B}(z^0, r), \psi, \delta) = \psi(z^0)$ for every nonempty ball $B(z^0, r)$. A compact K is globally (m, δ, ψ)-regular if it is globally (m, δ, ψ)-regular at every point of itself. A compact K is locally (m, δ, ψ)-regular if it is locally (m, δ, ψ)-regular at every point of itself.*

Note that global or local (m, δ, ψ)- regularity can only be defined for $\delta \in \Lambda(\psi, K)$. It is easy to see that any locally (m, δ, ψ)-regular point is globally (m, δ, ψ)-regular. We denote by $\Lambda_{reg} = \Lambda_{reg}(K, \psi)$ the set of numbers $\delta \subset \Lambda(K, \psi)$, for which K is globally regular, we denote by $\Lambda_{reg}^{loc} = \Lambda_{reg}^{loc}(K, \psi)$ the set of numbers $\delta \subset \Lambda$, for which K is locally regular. We see, $\Lambda_{reg}^{loc} \subset \Lambda_{reg} \subset \Lambda$.

Proposition 3.2. *Let $\delta_1, \delta_2 \in \Lambda(\psi, K)$ and $\delta_1 \leq \delta_2$. If a point z^0 is (m, δ_2, ψ)-regular, then it is (m, δ_1, ψ)-regular.*

The prove follows from the property 1° sec. 2. For the continuous function ψ holds

Theorem 3.1. *Let $\delta \in \Lambda(\psi, K)$ and the function $\psi(z)$ is to be continuous on K. Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (m, δ, ψ)-regular if and only if it is locally m-regular.*

Proof. Really, we use the inequality (5). If the point $z^0 \in K$ is not locally m-regular, i.e., if $V_m^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B: \quad z^0 \in B \subset \mathbb{C}^n$, then $V_m^*(z^0, K \cap \overline{B}_1) \geq \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (5)

$$V_m^*(z^0, K \cap \overline{B}_1, \psi(z^0), \delta) \geq (\delta - \inf_{z \in K \cap \overline{B}_1} \psi(z)) \cdot V_m^*(z^0, K \cap \overline{B}_1) + \inf_{z \in K \cap \overline{B}_1} \psi(z)$$

and

$$V_m^*(z^0, K \cap \overline{B}_1, \psi(z^0)) \geq \sigma \cdot \left(\delta - \inf_{z \in K \cap \overline{B}_1} \psi(z)\right) + \inf_{z \in K \cap \overline{B}_1} \psi(z) \tag{6}$$

Since $\psi(z)$ is continuous, then choosing the neighborhood B_1 small enough we can make the right part of (6) to be greater, than $\psi(z^0)$ i.e., $V_m^*(z, K \cap \overline{B}_1, \psi) > \psi(z^0)$ and the point z^0 is not locally (m, δ, ψ)-regular.

Reversing the roles of $V_m^*(z, K \cap \overline{B}_1, \psi)$ and $V_m^*(z, K \cap \overline{B}_1)$ from (5) we can prove the second part of the theorem: if the point $z^0 \in K$ is not locally (m, δ, ψ)-regular, then it is not locally m-regular. □

Corollary 3.1. *Let $\delta_1, \delta_2 \in \Lambda(\psi, K)$ and the function $\psi(z)$ is to be continuous on K. Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (m, δ_1, ψ)-regular if and only if it is locally (m, δ_2, ψ)-regular.*

Theorem 3.2. *Let $\psi(z)$ is continuous on K. If K is globally (m, δ, ψ)-regular i.e. if K is globally (m, δ, ψ)-regular at arbitrary point $z^0 \in K$, then $V_m^*(z, K, \psi, \delta) = V_m(z, K, \psi, \delta)$ and $V_m^*(z, K, \psi, \delta)$ continuous in $\mathbb{C}^n$.*

Proof. Let $\psi(z)$ be a function defined and continuous on K. It is well known that $\psi(z)$ can be extended continuously throughout K, i.e., there is a function $\Psi(z) \in C(\mathbb{C}^n)$ such that $\Psi(z)|_K = \psi(z)$ (see Whitney H. [12]). We use the approximation $u_j \downarrow V_m^*(z, K, \psi, \delta)$, where $u_j \in \mathcal{L}_m^\delta \cap C^\infty(\mathbb{C}^n)$. Since $V_m^*(z, K, \psi, \delta) \equiv \Psi(z),\ z \in K$, then for any $\varepsilon > 0$ there is an open set $\{z \in \mathbb{C}^n,\ V_m^*(z, K, \psi, \delta) < \Psi(z) + \varepsilon\}$ contained K. Therefore, by the Hartogs lemma, there exists $j_0 \in \mathbb{N}$ such that $u_j(z) < \Psi(z) + 2\varepsilon = \psi(z) + 2\varepsilon,\ \forall z \in K,\ j > j_0$. From here, $u_j - 2\varepsilon \in \mathcal{L}_m^\delta(\psi, K)$ and

$$u_j - 2\varepsilon \le V_m(z, K, \psi, \delta) \le V_m^*(z, K, \psi, \delta) \le u_j,\ j > j_0,\ z \in \mathbb{C}^n.$$

This means that the sequence u_j converges to $V_m^*(z, K, \psi, \delta)$ uniformly and $V_m^*(z, K, \psi, \delta) = V_m(z, K, \psi, \delta) \in C(\mathbb{C}^n)$. □

Proposition 3.3. *If $\delta_j \in \Lambda_{reg},\ \forall j \in \mathbb{N}$ and $\delta_j \uparrow \delta$ as $j \to \infty$, then $\delta \in \Lambda_{reg}$.*

Proof. In fact, since $\psi(z) = V_m^*(z, K, \psi, \delta_j), z \in K$, then

$$\psi(z) = V_m^*(z, K, \psi, \delta_j) = \delta_j \cdot V_m^*\left(z, K, \frac{\psi}{\delta_j}\right) \ge \delta_j \cdot V_m^*\left(z, K, \frac{\psi}{\delta}\right).$$

Therefore, $\forall j \in \mathbb{N}$ we have $\psi(z) \ge \delta_j \cdot V_m^*\left(z, K, \frac{\psi}{\delta}\right),\ z \in K$. Tending j to infinity, we get

$$\psi(z) \ge \delta \cdot V_m^*\left(z, K, \frac{\psi}{\delta}\right) = V_m^*(z, K, \psi, \delta),\ z \in K.$$

This means $\delta \in \Lambda_{reg}$. □

Corollary 3.2. *If $\Lambda = [\delta_0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & [\delta_0, \delta_1] \\ or & [\delta_0, \infty). \end{cases}$

Corollary 3.3. *If $\Lambda = (0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & (0, \delta_1] \\ or & (0, \infty). \end{cases}$

4 A property of (m, δ, ψ)-regularity

Further properties of the weighted (m, δ)-Green function are associated with m-thin sets.

Definition 4.1. *Let $E \subset \mathbb{C}^n$ and let E' be its limit point set. Then E is said to be m-thin at z^0 if either $z^0 \notin E'$ or $z^0 \in E'$ but there exists a neighbourhood U of z^0 and a function $u(z) \in sh_m(U)$ such that*

$$\overline{\lim_{\substack{z \to z^0 \\ z \in E \setminus \{z^0\}}}} u(z) < u(z^0)$$

So, if the set E is not thin at the point z^0, then for any m-subharmonic function $u(z)$ in the neighborhood of z^0

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, u(z) = \overline{\lim_{\substack{z\to z^0\\ z\in E}}}\, u(z) = u(z^0).$$

Note, that since $sh_m \subset sh$ then m-thin point z^0 is the same time classical thin point with respect to class sh functions. [5, 6, 7, 8].

Proposition 4.1. *If $E \subset \mathbb{C}^n$ is m-thin at a limit point z^0 of E, then there exists an m-subharmonic function $u \in \mathcal{L}_m$ such that*

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, u(z) < u(z^0).$$

Proof. Let $v(z) \in sh_m(B(z^0, r))$ and $\overline{\lim\limits_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, v(z) < v(z^0)$. Without loss of generality we may assume that $v < 0$ in the ball $B(z^0, r)$. Then the function

$$u(z) = \begin{cases} \max\left\{ v(z), -\dfrac{1}{|z-z^0|^{2\left(\frac{n}{m}-1\right)}} \right\}, & z \in B(z^0, r) \\ -\dfrac{1}{|z-z^0|^{2\left(\frac{n}{m}-1\right)}}, & z \notin B(z^0, r). \end{cases}$$

belongs to $\mathcal{L}_m$ and

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, v(z) = \overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, u(z) < u(z^0) = v(z^0).$$

□

Theorem 4.1. *If z^0 is an m-thin point of K, then z^0 is locally (m, δ, ψ)-irregular point of K. Here the function $\psi \in L^\infty(K)$ and $\delta \in \Lambda$.*

◁ Let K be m-thin at the point $z^0 \in K$. Then, according to Proposition 4.1, there exists a function $u(z) \in \mathcal{L}_m^\delta$ such that

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}}\, u(z) < u(z^0)$$

Without loss of generality, we can assume $u(z^0) > 0$ and find a ball $B(z^0, r)$ such that

$$\begin{cases} u(z) \le \inf\limits_{z\in K} \psi(z) - \psi(z^0) \text{ for } z \in K \cap B \setminus \{z^0\}, \\ u(z^0) > 0. \end{cases}$$

Put $w(z) = u(z) + \psi(z^0)$. It is easy to see that $w(z) \in L_m^\delta(\psi, K \cap B \setminus \{z^0\})$, because for $z \in K \cap B \setminus \{z^0\}$

$$w(z) = u(z) + \psi(z^0) \le \inf_{z\in K} \psi(z) - \psi(z^0) + \psi(z^0) = \inf_{z\in K} \psi(z) \le \psi(z).$$

Consequently,

$$w(z) \leq V_m^*(z, K \cap B \setminus \{z^0\}, \psi, \delta) = V_m^*(z, K \cap B, \psi, \delta), \ \forall z \in \mathbb{C}^n.$$

From here

$$w(z^0) \leq V_m^*(z^0, K \cap B, \psi, \delta).$$

In the other hand

$$w(z^0) = u(z^0) + \psi(z^0) > \psi(z^0).$$

Therefore

$$\psi(z^0) < w(z^0) \leq V_m^*(z^0, K \cap B, \psi, \delta).$$

Hence, the point z^0 is a locally (m, δ, ψ)-irregular point of the compact set K. $\triangleright$

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