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Muzaffar Eshimbetov
Institute of Mathematics named after V.I. Romanovsky, Tashkent, Uzbekistan, mr.eshimbetov@gmail.com

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ON EXTENSIONS AND RESTRICTIONS OF τ-SMOOTH AND τ-MAXITIVE IDEMPOTENT MEASURES

Eshimbetov M.R.
Institute of Mathematics named after V.I. Romanovsky, Tashkent, Uzbekistan
e-mail: mr.eshimbetov@gmail.com

Abstract

In the paper we investigate maps between idempotent measures spaces, τ-maxitive idempotent measures and their extensions and restrictions. For an idempotent measure we prove that its extension is τ-maxitive if and only if its restriction is τ-maxitive.

Keywords: *Luzin measurable function, τ-smooth idempotent measure, τ-maxitive idempotent measure.*

Mathematics Subject Classification (2010): *28C10, 46G12, 58D20.*

Introduction

Idempotent mathematics is a branch of mathematical sciences, rapidly developing and gaining popularity over the last four decades. An important stage of development of the subject was presented in the book "Idempotency" [4] edited by J. Gunawardena. This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994.

The next stage of development of idempotent and tropical mathematics was presented in the book Idempotent Mathematics and Mathematical Physics edited by G. L. Litvinov and V. P. Maslov. The book arose out of the international workshop that was held in Vienna, Austria, in February 2003.

Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i. e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra $\mathbb{R}_{\max}$ [5], [6], [8].

Many authors (S. C. Kleene, S. N. N. Pandit, N. N. Vorobjev, B. A. Carré, R. A. Cuninghame-Green, K. Zimmermann, U. Zimmermann, M. Gondran, F. L. Baccelli, G. Cohen, S. Gaubert, G. J. Olsder, J.-P. Quadrat, V. N. Kolokoltsov, A. A. Zaitov and others) used idempotent semirings and matrices over these semirings for solving some applied problems in computer science and discrete mathematics, starting from the classical paper by S. C. Kleene [9], [10], [11], [12].

In the paper we investigate maps between idempotent measures spaces, τ-maxitive idempotent measures and their extensions and restrictions. For an idempotent measure we prove that its extension is τ-maxitive if and only if its restriction is τ-maxitive.

1 Preliminaries

Let Ω be a set and $\mathcal{E}$ is a set system of subsets of Ω, which contains $\varnothing$. Let $\mathrm{P}(\Omega)$ denote the power set of Ω and $\overline{\mathbb{R}}_+ = [0, +\infty)\bigcup\{+\infty\} = [0, +\infty]$. The symbol Φ denotes the directed sets, and Δ an arbitrary index sets.

Definition 1.1. *[7]. A set function $\mu\colon \mathrm{P}(\Omega) \to \overline{\mathbb{R}}_+$ is an idempotent measure on Ω if the following conditions hold*

1) $\mu(\varnothing) = 0$*;*

2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ *for any* $A, B \in \mathrm{P}(\Omega)$*;*

3) $\mu\left(\underset{\phi\in\Phi}{\cup} A_\phi\right) = \sup\limits_{\phi\in\Phi}\{\mu(A_\phi)\}$ *for an arbitrary increasing net* $\{A_\phi, \phi \in \Phi\}$ *of subsets of* Ω.

The set of all idempotent measure on Ω we denote by $M(\Omega)$. If $\mu(\Omega) = 1$, the idempotent measure μ is called an idempotent probability measure on Ω.

Definition 1.2. *[7]. If, in addition to the conditions of definition 1.1, the set function μ has the following property*

4) $\mu\left(\underset{\phi\in\Phi}{\cap} F_\phi\right) = \inf\limits_{\phi\in\Phi}\{\mu(F_\phi)\}$ *for an arbitrary decreasing net* $\{F_\phi : \phi \in \Phi\}$ *of elements of* $\mathcal{E}$,

then idempotent measure μ is called a τ-smooth idempotent measure with respect to $\mathcal{E}$ on Ω or, for short, is an $\mathcal{E}$-idempotent measure.

Remark 1. *Every idempotent measure μ is increasing, i. e. if $A \subset B$, then $\mu(A) \leq \mu(B)$.*

Definition 1.3. *[7]. 1) A set function $\mu\colon \mathcal{E} \to \overline{\mathbb{R}}_+$ is called maxitive on $\mathcal{E}$, if*

$$\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$$

for any $A \in \mathcal{E}$ and $B \in \mathcal{E}$ such that $A \cup B \in \mathcal{E}$.

2) A set function $\mu\colon \mathcal{E} \to \overline{\mathbb{R}}_+$ is called τ-maxitive on $\mathcal{E}$, if

$$\mu\left(\underset{\alpha\in\Delta}{\cup} A_\alpha\right) = \sup_{\alpha\in\Delta}\{\mu(A_\alpha)\}$$

for every collection of sets A_α, $\alpha \in \Delta$, such that $\underset{\alpha\in\Delta}{\cup} A_\alpha \in \mathcal{E}$.

The set of all τ-maxitive on $\mathcal{E}$ idempotent measures we denote by $M^{\tau}(\Omega)$.

Definition 1.4. *[7]. Let $\mathcal{A}$ be a collection of subsets of a set Ω. Then $\mathcal{A}$ is called a τ-algebra on Ω, if the following conditions hold:*

1) $\varnothing \in \mathcal{A}$;

2) *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$, where $A^c = \Omega \backslash A$ (i. e., $\mathcal{A}$ is closed under the formation of complement);*

3) *If $\mathcal{A}_0 \subset \mathcal{A}$ then $\cup \mathcal{A}_0 \in \mathcal{A}$ (i. e., $\mathcal{A}$ is closed under the formation of arbitrary union).*

The elements of $\mathcal{A}$ are referred as $\mathcal{A}$-measurable subsets of Ω.

Remark 2. *[2]. Let $\mathcal{A}$ be a τ-algebra. Then the following relations are true:*
1) $A \cap B \in \mathcal{A}$ for an arbitrary A, $B \in \mathcal{A}$;
2) $A \setminus B \in \mathcal{A}$ and $B \backslash A = B \cap A^c \in \mathcal{A}$ for an arbitrary A, $B \in \mathcal{A}$;
3) $A \triangle B \in \mathcal{A}$ for an arbitrary A, $B \in \mathcal{A}$.

Let a set function μ be an $\mathcal{E}$-idempotent measure.

Definition 1.5. *[7]. The system $\mathcal{T}$ of subsets of Ω is called tightening for μ if $T \cap F \in \mathcal{E}$ for $T \in \mathcal{T}$ and $F \in \mathcal{E}$, and also for each $\varepsilon > 0$ there exists $T \in \mathcal{T}$ such that $\mu(T^c) \leq \varepsilon$. In this case, μ is called tight relative to $\mathcal{T}$ or, for short, $\mathcal{T}$-tight.*

Let $(\Omega_1, \mathcal{E}_1)$ and $(\Omega_2, \mathcal{E}_2)$ be τ-measurable spaces. Recall, a function $f\colon \Omega_1 \to \Omega_2$ is an $\mathcal{E}_1/\mathcal{E}_2$-measurable, if $f^{-1}(A) \in \mathcal{E}_1$ for each $A \in \mathcal{E}_2$.

Let the system $\mathcal{T}$ be a contraction for the $\mathcal{E}$ - idempotent measure μ.

Definition 1.6. *[1]. The function $f\colon \Omega_1 \to \Omega_2$ is called $(\mathcal{E}_1, \mathcal{T})/\mathcal{E}_2$-measurable in the sense of Luzin if the restriction $f|_T$ of the function f to an arbitrary $T \in \mathcal{T}$ is $\mathcal{E}_T/\mathcal{E}_2$-measurable, where $\mathcal{E}_T = \{T \cap F : F \in \mathcal{E}_1\}$.*

2 Main Part

Consider any sets Ω_1 and Ω_2, and a system $\mathcal{E}_2$ of subsets of Ω_2. For a map $f\colon \Omega_1 \to \Omega_2$ we put

$$f^{-1}(\mathcal{E}_2) = \{f^{-1}(B) : B \in \mathcal{E}_2\}.$$

Lemma 2.1. *[7]. If a system $\mathcal{E}_2$ is a τ-algebra on Ω_2 then the system $f^{-1}(\mathcal{E}_2)$ is also a τ-algebra on Ω_1.*

For a map $f\colon \Omega_1 \to \Omega_2$ let $M(f)$ denote a map from the set $M(\Omega_1)$ into the set $M(\Omega_2)$ defined by the rule

$$M(f)(\mu)(B) = \mu(f^{-1}(B)), \qquad B \subset \Omega_2.$$

Lemma 2.2. *[7]. For a map $f\colon \Omega_1 \to \Omega_2$ and an idempotent measure μ on Ω_1 the set function $M(f)(\mu)\colon \mathrm{P}(\Omega_2) \to \overline{\mathbb{R}}_+$ is an idempotent measure on Ω_2.*

Let Ω be a set, Ω_0 a subset of Ω and $\mathcal{E}$ a τ-algebra on Ω. We put

$$\mathcal{E}_0 = \{B \cap \Omega_0 : B \in \mathcal{E}\}.$$

Then it is easy to see that $\mathcal{E}_0$ is a τ-algebra on Ω_0. Note that, in general, the system $\mathcal{E}_0$ should not be a subsystem of $\mathcal{E}$. Clearly, if $\Omega_0 \in \mathcal{E}$ then it immediately yields that $\mathcal{E}_0 \subset \mathcal{E}$.

Assume an idempotent measure $\mu\colon \mathrm{P}(\Omega_0) \to \overline{\mathbb{R}}_+$ is given [2]. By the rule

$$\mathrm{e}_{\Omega}^{\Omega_0}(\mu)(B) = \mu(B \cap \Omega_0), \qquad B \subset \Omega,$$

we determine its extension $\mathrm{e}_{\Omega}^{\Omega_0}(\mu)\colon \mathrm{P}(\Omega) \to \overline{\mathbb{R}}_+$.

Let Ω_2' be a subset of Ω_2, $\Omega_1 = f^{-1}(\Omega_2')$ and $\mu \in M(\Omega_1)$. Consider a set function $\mathrm{e}_{\Omega_2}^{\Omega_2'}(M(f)(\mu)) : \mathrm{P}(\Omega_2) \to \overline{\mathbb{R}}_+$ defined as

$$\mathrm{e}_{\Omega_2}^{\Omega_2'}(M(f)(\mu))(A) = \mu\left(f^{-1}(A \cap \Omega_2')\right), \qquad A \subset \Omega_2.$$

Let

$$\mathcal{K} = \{B \cap \Omega_2' : B \in \mathcal{E}_2\}.$$

Then

$$f^{-1}(\mathcal{K}) = f^{-1}(\mathcal{E}_2).$$

Indeed, if $T \subset \Omega_2$, $T \cap \Omega_2' = \varnothing$ then

$$f^{-1}(T) = f^{-1}(T) \cap \Omega_1 = f^{-1}(T) \cap f^{-1}(\Omega_2') = f^{-1}(T \cap \Omega_2') = f^{-1}(\varnothing) = \varnothing.$$

If $T \subset \Omega_2,\ T \cap \Omega_2' \neq \varnothing$ then $f^{-1}(T) = f^{-1}(T \cap \Omega_2')$.

Let the set function μ be an $f^{-1}(\mathcal{E}_2)$-idempotent measure. Clearly, the system $f^{-1}(\mathcal{E}_2)$ is a tightening for μ, because for each $\varepsilon > 0$ there exists $T \in f^{-1}(\mathcal{E}_2)$ such that $\mu(T^c) \leq \varepsilon$. For example, $T = \Omega_1 \in f^{-1}(\mathcal{E}_2)$. Obviously, the function $f\colon \Omega_1 \to \Omega_2$ be a $(\mathcal{E}_1,\, f^{-1}(\mathcal{E}_2))/\mathcal{E}_2$-measurable in the sense of Luzin. Really, the restriction $f|_T$ of the function f to an arbitrary $T \in f^{-1}(\mathcal{E}_2)$ is $\mathcal{E}_T/\mathcal{E}_2$-measurable, where $\mathcal{E}_T = \{T \cap F : F \in \mathcal{E}_1\}$.

Proposition 1. *If an idempotent measure μ is τ-smooth with respect to $f^{-1}(\mathcal{E}_2)$ on Ω_1 then the set function $\mathrm{e}_{\Omega_2}^{\Omega_2'}(M(f)(\mu))$ is an idempotent measure, moreover it is τ-smooth with respect to $\mathcal{E}_2$ on Ω_2.*

Proof. We should check following conditions:

1. $\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(\varnothing) = \mu(f^{-1}(\varnothing)) = \mu(\varnothing) = 0;$
2. Take any $A,\ B \in \mathrm{P}(\Omega_2)$. Then

$$\begin{aligned}\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(A\cup B) =&\mu(f^{-1}((A\cup B)\cap\Omega_2^{'})) = \\ =&\mu(f^{-1}(A\cap\Omega_2^{'})\cup f^{-1}(B\cap\Omega_2^{'})) = \\ =&\max\{\mu(f^{-1}(A\cap\Omega_2^{'})),\ \mu(f^{-1}((B\cap\Omega_2^{'}))\} = \\ =&\max\{\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(A),\ \mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(B)\};\end{aligned}$$

3. Take an increasing net $\{A_\phi,\ \phi\in\Phi\}$ of subsets of Ω_2. Then $\left\{A_\phi\cap\Omega_2^{'},\ \phi\in\Phi\right\}$ is an increasing net as well. Then the system $\left\{f^{-1}(A_\phi\cap\Omega_2^{'}),\ \phi\in\Phi\right\}$ is an increasing net of subsets of $\Omega_1^{'}$. Therefore, it takes place the equality

$$\mu\left(\bigcup_{\phi\in\Phi}\left(f^{-1}\left(A_\phi\cap\Omega_2^{'}\right)\right)\right) = \sup_{\phi\in\Phi}\{\mu\left(f^{-1}(A_\phi\cap\Omega_2^{'})\right)\}.$$

Hence,

$$\begin{aligned}\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)\left(\bigcup_{\phi\in\Phi}A_\phi\right) &= \mu\left(f^{-1}\left(\left(\bigcup_{\phi\in\Phi}A_\phi\right)\cap\Omega_2^{'}\right)\right) = \\ =&\mu\left(f^{-1}\left(\bigcup_{\phi\in\Phi}\left(A_\phi\cap\Omega_2^{'}\right)\right)\right) = \mu\left(\bigcup_{\phi\in\Phi}\left(f^{-1}\left(A_\phi\cap\Omega_2^{'}\right)\right)\right) = \\ =&\sup_{\phi\in\Phi}\{\mu\left(f^{-1}(A_\phi\cap\Omega_2^{'})\right)\} = \sup_{\phi\in\Phi}\{\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(A_\phi)\};\end{aligned}$$

4. Take a decreasing net $\{F_\phi,\ \phi\in\Phi\}$ of elements of $\mathcal{E}_2$. Then $\left\{F_\phi\cap\Omega_2^{'},\ \phi\in\Phi\right\}$ is a decreasing net as well. As a result we have that the system $\left\{f^{-1}(F_\phi\cap\Omega_2^{'}),\ \phi\in\Phi\right\}$ is a decreasing net in $f^{-1}(\mathcal{E}_2)$. Since the idempotent measure μ is τ-smooth with respect to $f^{-1}(\mathcal{E}_2)$ then the following equality is true

$$\mu\left(\bigcap_{\phi\in\Phi}\left(f^{-1}\left(F_\phi\cap\Omega_2^{'}\right)\right)\right) = \inf_{\phi\in\Phi}\left\{\mu\left(f^{-1}\left(F_\phi\cap\Omega_2^{'}\right)\right)\right\}.$$

Hence

$$\begin{aligned}\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)\left(\bigcap_{\phi\in\Phi}F_\phi\right) &= \mu\left(f^{-1}\left(\left(\bigcap_{\phi\in\Phi}F_\phi\right)\cap\Omega_2^{'}\right)\right) = \\ =&\mu\left(f^{-1}\left(\bigcap_{\phi\in\Phi}\left(F_\phi\cap\Omega_2^{'}\right)\right)\right) = \mu\left(\bigcap_{\phi\in\Phi}\left(f^{-1}\left(F_\phi\cap\Omega_2^{'}\right)\right)\right) = \\ =&\inf_{\phi\in\Phi}\left\{\mu\left(f^{-1}\left(F_\phi\cap\Omega_2^{'}\right)\right)\right\} = \inf_{\phi\in\Phi}\left\{\mathrm{e}_{\Omega_2}^{\Omega_2'}\left(M(f)(\mu)\right)(F_\phi)\right\}.\end{aligned}$$

Proposition 1 is proved.

□

Proposition 2. *If an idempotent measure μ is τ-maxitive on $f^{-1}(\mathcal{E}_2)$ then the idempotent measure $e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))$ also is τ-maxitive on $\mathcal{E}_2$.*

Proof. Since an idempotent measure μ is τ-maxitive with respect to $f^{-1}(\mathcal{E}_2)$ then there exists a system of sets $f^{-1}(B_j) \in f^{-1}(\mathcal{E}_2)$, $j \in J$, such that $\underset{j\in J}{\cup} f^{-1}(B_j) \in f^{-1}(\mathcal{E}_2)$, and

$$\mu\left(\underset{j\in J}{\cup} f^{-1}(B_j)\right) = \sup_{j\in J}\{\mu(f^{-1}(B_j))\}.$$

It is clear, there exists a system of sets $B_j \in \mathcal{E}_2$, $j \in J$, such that $\underset{j\in J}{\cup} B_j \in \mathcal{E}_2$. For this system by definition of τ-maxitivity we have

$$e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))\left(\underset{j\in J}{\cup} B_j\right) = \mu\left(f^{-1}\left(\underset{j\in J}{\cup} B_j\right)\right) = \mu\left(\underset{j\in J}{\cup} f^{-1}(B_j)\right) =$$
$$= \sup_{j\in J}\{\mu(f^{-1}(B_j))\} = \sup_{j\in J}\{e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))(B_j)\}.$$

So, the idempotent measure $e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))$ is τ-maxitive with respect to $\mathcal{E}_2$. Proposition 2 is proved.

□

Proposition 3. *If $e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))$ is a τ-maxitive idempotent measure on $\mathcal{E}_2$ then μ is also a τ-maxitive idempotent measure on $f^{-1}(\mathcal{E}_2)$.*

Proof. Take a system of sets $B_j \in \mathcal{E}_2$, $j \in J$, such that $\underset{j\in J}{\cup} B_j \in \mathcal{E}_2$. By condition we have

$$e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))\left(\underset{j\in J}{\cup} B_j\right) = \sup_{j\in J}\{e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))(B_j)\}.$$

Obviously, $B_j \cap \Omega_2' \in \mathcal{K}$ for every $j \in J$ and $f^{-1}\left(B_j \cap \Omega_2'\right) \in f^{-1}(\mathcal{K}) = f^{-1}(\mathcal{E}_2)$, $j \in J$, and the inclusion $\underset{j\in J}{\cup} B_j \in \mathcal{E}_2$ implies $\underset{j\in J}{\cup}\left(B_j \cap \Omega_2'\right) \in \mathcal{K}$. Therefore, we obtain $\underset{j\in J}{\cup} f^{-1}\left(B_j \cap \Omega_2'\right) \in f^{-1}(\mathcal{E}_2)$ for every $j \in J$.

Now from the definition of $e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))$ one has

$$e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))\left(\underset{j\in J}{\cup} B_j\right) = \mu\left(f^{-1}\left(\underset{j\in J}{\cup} B_j \cap \Omega_2'\right)\right) =$$
$$= \mu\left(f^{-1}\left(\underset{j\in J}{\cup}\left(B_j \cap \Omega_2'\right)\right)\right) = \mu\left(\underset{j\in J}{\cup}\left(f^{-1}\left(B_j \cap \Omega_2'\right)\right)\right) =$$
$$= \sup_{j\in J}\{e_{\Omega_2}^{\Omega_2'}(M(f)(\mu))(B_j)\} = \sup_{j\in J}\{\mu(f^{-1}(B_j \cap \Omega_2'))\}.$$

So, we obtain the following equality

$$\mu\left(\underset{j\in J}{\cup}\left(f^{-1}\left(B_j\cap\Omega_2'\right)\right)\right)=\sup_{j\in J}\{\mu(f^{-1}(B_j\cap\Omega_2'))\}$$

for a collection of sets $f^{-1}\left(B_j\cap\Omega_2'\right)\in f^{-1}\left(\mathcal{E}_2\right)$, $j\in J$, such that $\underset{j\in J}{\cup}f^{-1}\left(B_j\cap\Omega_2'\right)\in f^{-1}\left(\mathcal{E}_2\right)$.

Proposition 3 is proved.

$\square$

Propositions 2 and 3 provide that the following property is true.

Theorem 1. *$\mu\in M^\tau(\Omega_1)$ if and only if $\mathrm{e}_{\Omega_2}^{\Omega_2'}(M(f)(\mu))\in M^\tau(\Omega_2)$.*

Let an idempotent measure $\mu\in M(\Omega_1)$ be given. A set function

$$\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right):\ \mathrm{P}(\Omega_2')\to\overline{\mathbb{R}}_+$$

we define by

$$\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)(A)=\inf\{\mu\left(f^{-1}(C)\right):\ C\in\mathcal{E}_2,\ C\cap f(\Omega_1)=A\},$$

where $A\subset\Omega_2'=f(\Omega_1)$.

Proposition 4. *If μ is a τ-smooth idempotent measure according to the system $f^{-1}(\mathcal{E}_2)$ on Ω_1 then a set function $\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)$ is also a τ-smooth idempotent measure according to the system $\mathcal{K}$ on Ω_2' (remind, here $\mathcal{K}=\{B\cap\Omega_2':\ B\in\mathcal{E}_2\}$).*

Proof. We should check the conditions, stated in Definitions 1.1 and 1.2 for a set function $\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)$.

1) $\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)(\varnothing)=\inf\{\mu(f^{-1}(C)):\ C\in\mathcal{E}_2,\ C\cap f(\Omega_1)=\varnothing\}=\mu(\varnothing)=0$;

2) Take any sets $A,\ B\in\mathrm{P}(\Omega_2')$. Then

$$\begin{aligned}
&\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)(A\cup B)=\\
&=\inf\{\mu(f^{-1}(C_A\cup C_B)):\ C_A,\ C_B\in\mathcal{E}_2,\ C_A\cap f(\Omega_1)=A,\ C_B\cap f(\Omega_1)=B\}=\\
&=\inf\{\max\{\mu(f^{-1}(C_A)),\ \mu(f^{-1}(C_B))\}:\\
&\qquad\qquad C_A,\ C_B\in\mathcal{E}_2,\ C_A\cap f(\Omega_1)=A,\ C_B\cap f(\Omega_1)=B\}=\\
&=\max\{\inf\{\mu(f^{-1}(C_A)):\ C_A\in\mathcal{E}_2,\ C_A\cap f(\Omega_1)=A\},\\
&\qquad\qquad\inf\{\mu(f^{-1}(C_B)):\ C_B\in\mathcal{E}_2,\ C_B\cap f(\Omega_1)=B\}\}=\\
&=\max\{\mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)(A),\ \mathrm{r}_{\Omega_2'}^{\Omega_2}\left(M(f)(\mu)\right)(B)\};
\end{aligned}$$

3) Take an increasing net $\{A_\phi,\, \phi\in\Phi\}$ of subsets of Ω_2'. Then $\{f^{-1}(C_\phi),\, \phi\in\Phi\}$ is an increasing net of subsets of Ω_1. For this net we have

$$\begin{aligned}
\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\,(\underset{\phi}{\cup}A_\phi) &= \inf\{\mu(f^{-1}(\underset{\phi}{\cup}C_\phi)) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\} = \\
&= \inf\{\mu(\underset{\phi}{\cup}(f^{-1}(C_\phi))) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\} = \\
&= \inf\{\sup_\phi\{\mu(f^{-1}(C_\phi))\} : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\} = \\
&= \sup_\phi\{\inf\{\mu(f^{-1}(C_\phi)) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\}\} = \\
&= \sup_\phi\{\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\,(A_\phi)\}.
\end{aligned}$$

4) Take a decreasing net $\{F_\phi,\, \phi\in\Phi\}$ of elements of $\mathcal{K}$. Then $\{f^{-1}(C_\phi),\, \phi\in\Phi\}$ is a decreasing net of elements of $f^{-1}(\mathcal{E}_2)$. For this net the following take place

$$\begin{aligned}
\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\,(\underset{\phi}{\cap}F_\phi) &= \inf\{\mu(f^{-1}(\underset{\phi}{\cap}C_\phi)) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\} = \\
&= \inf\{\mu(\underset{\phi}{\cap}(f^{-1}(C_\phi))) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\} = \\
&= \inf\{\inf_\phi\{\mu(f^{-1}(C_\phi))\} : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=F_\phi,\ \phi\in\Phi\} = \\
&= \inf_\phi\{\inf\{\mu(f^{-1}(C_\phi)) : C_\phi\in\mathcal{E}_2,\ C_\phi\cap f(\Omega_1)=A_\phi,\ \phi\in\Phi\}\} = \\
&= \inf_\phi\{\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\,(F_\phi)\}.
\end{aligned}$$

Proposition 4 is proved.

□

Propositions 1 and 4 give the following result which is one of the main achievements the paper.

Theorem 2. *Let $f\colon\ \Omega_1\to\Omega_2$ be a map, $f(\Omega_1)=\Omega_2'\subset\Omega_2$. Then for any $\mu\in M(\Omega_1)$ one has $\mathrm{e}_{\Omega_2}^{\Omega_2'}(M(f)(\mu))\in M(\Omega_2)$ and, conversely, for any $\mu\in M(\Omega_1)$ one has $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\in M(f(\Omega_1))$.*

Proposition 5. *If μ is a τ-maxitive idempotent measure on $f^{-1}(\mathcal{E}_2)$ then a set function $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))$ is also a τ-maxitive idempotent measure on $\mathcal{K}$.*

Proof. Consider a collection of sets $B_j\in\mathcal{K}$, $j\in J$. Of course, for every $j\in J$ there is a set $C_j\in\mathcal{E}_2$ such that $C_j\cap\ f(\Omega_1)=B_j$. Then $\underset{j\in J}{\cup}B_j=\underset{j\in J}{\cup}(C_j\cap f(\Omega_1))=\left(\underset{j\in J}{\cup}C_j\right)\cap\ f(\Omega_1)\in\mathcal{K}$. According to the condition of τ-maxitivity of the idempotent measure μ on the system $f^{-1}(\mathcal{E}_2)$ we have:

$$\begin{aligned}
\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\left(\underset{j\in J}{\cup}B_j\right) &= \inf\{\mu\left(f^{-1}\left(\underset{j\in J}{\cup}C_j\right)\right) : C_j\in\mathcal{E}_2,\, C_j\cap f(\Omega_1)=B_j\} = \\
&= \inf\{\mu\left(\underset{j\in J}{\cup}f^{-1}(C_j)\right) : C_j\in\mathcal{E}_2,\, C_j\cap f(\Omega_1)=B_j\} = \\
&= \inf\{\sup_j\{\mu(f^{-1}(C_j)) : C_j\in\mathcal{E}_2,\, C_j\cap f(\Omega_1)=B_j\}\} = \\
&= \sup_j\{\inf\{\mu(f^{-1}(C_j)) : C_j\in\mathcal{E}_2,\, C_j\cap f(\Omega_1)=B_j\}\} = \\
&= \sup_{j\in J}\{\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))(B_j)\}.
\end{aligned}$$

So, $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))$ is τ-maxitive on the system $\mathcal{K}$.

Proposition 5 is proved.

□

Proposition 6. *If $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))$ is a τ-maxitive idempotent measure on $\mathcal{K}$ then μ is also a τ-maxitive idempotent measure on $f^{-1}(\mathcal{E}_2)$.*

Proof. Take a collection of sets $B_j\in\mathcal{K}$, $j\in J$ such that $\underset{j\in J}{\cup}B_j\in\mathcal{K}$. No loosing generality we may suppose that the collection $\{B_j : j\in J\}$ is increasing.

Indeed, take any system $\{B_j : j\in J\}$. Denote $B_1'=B_1$ and $B_j'=\underset{i<j}{\cup}B_i$, $j>1$. Clearly, $\underset{j\in J}{\cup}B_j'=\underset{j\in J}{\cup}B_j$ and the obtained system $\{B_j',\, j\in J\}$ is increasing as well.

With respect to the condition of τ-maxitivity of the idempotent measure $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))$ on the system $\mathcal{K}$ one has

$$\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))\left(\underset{j\in J}{\cup}B_j\right)=\sup_{j\in J}\{\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))(B_j)\}.$$

The definition of $\mathrm{r}_{\Omega_2'}^{\Omega_2}(M(f)(\mu))$ provided that the following holds:

$$\begin{aligned}
&\inf\left\{\mu\left(f^{-1}\left(\underset{j\in J}{\cup}C_j\right)\right) : C_j\in\mathcal{E}_2,\, C_j\cap\ f(\Omega_1)=B_j\right\} = \\
&=\inf\left\{\mu\left(\underset{j\in J}{\cup}f^{-1}(C_j)\right) : C_j\in\mathcal{E}_2,\, C_j\cap\ f(\Omega_1)=B_j\right\} = \\
&=\sup_j\left\{\inf\{\mu\left(f^{-1}(C_j)\right) : C_j\in\mathcal{E}_2,\, C_j\cap\ f(\Omega_1)=B_j\}\right\}.
\end{aligned}$$

In this case we can suppose that a collection $\{C_j\}$ is also increasing as well and a collection of sets $f^{-1}(C_j)\in f^{-1}(\mathcal{E}_2)$, $j\in J$ such that $\underset{j\in J}{\cup}f^{-1}(C_j)\in f^{-1}(\mathcal{E}_2)$.

Therefore, a net $\{f^{-1}(C_j)\}$ is increasing as well. Hence, we have

$$\inf\left\{\mu\left(\underset{j\in J}{\cup} f^{-1}(C_j)\right) : C_j\in\mathcal{E}_2,\, C_j\cap\ f(\Omega_1)=B_j\right\}=$$
$$=\inf\left\{\sup_j\{\mu\left(f^{-1}(C_j)\right) : C_j\in\mathcal{E}_2,\, C_j\cap\ f(\Omega_1)=B_j\}\right\}.$$

Suppose a system $\{C_j,\, j\in J\}$ is increasing. Then $\overset{j-1}{\underset{i=1}{\cup}} C_i\subset C_j$ for every $j\in J$. From here $\overset{j-1}{\underset{i=1}{\cup}} f^{-1}(C_i)\subset f^{-1}(C_j)$ for every $j\in J$. Hence $\mu\left(\overset{j-1}{\underset{i=1}{\cup}} f^{-1}(C_i)\right)\leq \mu(f^{-1}(C_j))$, $j\in J$. From the last inequality we have

$$\mu\left(\underset{j\in J}{\cup} f^{-1}(C_j)\right)\leq \sup_{j\in J}\{\mu(f^{-1}(C_j))\}.$$

On the other hand $f^{-1}(C_j)\subset \underset{j\in J}{\cup} f^{-1}(C_j)$ implies $\mu(\underset{j\in J}{\cup} f^{-1}(C_j))\geq \mu(f^{-1}(C_j))$ for every $j\in J$. This implies $\mu(\underset{j\in J}{\cup} f^{-1}(C_j))\geq \sup_{j\in J}\{\mu(f^{-1}(C_j))\}$.

As the result we obtain the following equality

$$\mu\left(\underset{j\in J}{\cup} f^{-1}(C_j)\right)=\sup_{j\in J}\{\mu\left(f^{-1}(C_j)\right)\}.$$

Thus μ is τ-maxitive in means of the system $f^{-1}(\mathcal{E}_2)$.

Proposition 6 is proved.

□

Propositions 5 and 6 imply the following

Theorem 3. *$\mu\in M^\tau(\Omega_1)$ if and only if $\mathrm{r}^{\Omega_2}_{\Omega_2'}(M(f)(\mu))\in M^\tau(f(\Omega_1))$.*

Theorems 2 and 3 yield the following statement.

Corollary 1. *$\mathrm{e}^{\Omega_2'}_{\Omega_2}(M(f)(\mu))\in M^\tau(\Omega_2)$ if and only if $\mathrm{r}^{\Omega_2}_{\Omega_2'}(M(f)(\mu))\in M^\tau(f(\Omega_1))$.*

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