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Botir Zakirov
Tashkent State Transport University, botirzakirov@list.ru

Khabibulla Umarov
Gulistan State University, umarovhr@mail.ru

DUALITY FOR L^1-SPACES ASSOCIATED WITH THE MAHARAM MEASURE

Zakirov B.S.[1], Umarov Kh.R.[2]
[1] *Tashkent State Transport University, Tashkent, Uzbekistan*
[2] *Gulistan State University, Gulistan, Uzbekistan*
e-mail: botirzakirov@list.ru, umarovhr@mail.ru

Abstract

Dual space for the Banach-Kantorovich space $L^1(m)$ of all functions integrable with respect to a Maharam measure m is described and its pre-dual space is constructed.

Keywords: *vector-valued integration, Maharam measure, Banach-Kantorovich space.*

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Introduction

It is known that for any σ-additive measure with values in a *bo*-complete lattice normed space, a Lebesgue-type integral of numerical functions can be constructed. This is fairly straightforward and all simple properties of the resultant integral as well as analog of the is Lebesgue convergence results are easily deduced (see, for example, [1], [9]).

The question of when the corresponding space of integrable elements is a Banach-Kantorovich space is also completely solved.

In this paper, we consider a strongly positive measure m defined on a complete Boolean algebra B, which takes on a value in the algebra L^0 of all real measurable functions. The integral with respect to the measure m is constructed for the elements of the algebra $L^0(B) := C_\infty(Q(B))$ of all continuous functions $x : Q(B) \to \overline{\mathbb{R}} = [-\infty, +\infty]$, defined on the Stone compact $Q(B)$ corresponding to the Boolean algebra B, such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$.

We study the order and topological properties of the space $L^1(B, m) \subset L^0(B)$ of all functions integrable with respect to the Maharam measure m. Using a Boolean subalgebra $A \subset B$, a set $L^\infty(B, A)$ of all A-bounded elements from $L^0(B)$ is introduced.

It is shown that $L^\infty(B, A)$ is a subalgebra in $L^0(B)$ with a L^0-valued norm, endowing $L^\infty(B, A)$ the structure of the Banach-Kantarovich space. It is proved that the space $L^\infty(B, A)$ is isometrically isomorphic to the dual space $L^1(B, m)^*$ to $L^1(B, m)$, and the space $L^1(B, m)$ is isometric closed subspace in $L^\infty(B, A)^*$ consisting of normal L^0-bounded linear mappings acting from $L^\infty(B, A)$ to L^0.

We use terminology and notations of the theory of Boolean algebras from [2], the theory of vector lattices from [3], the theory of vector integration, and the theory of Banach - Kantorovich spaces from [1].

1 Preliminaries

Let X be a vector space over the field $\mathbb{R}$ of real numbers and let F be an order complete vector lattice. A mapping $\|\cdot\| : X \to F$ is said to be a vector (F-valued) norm if it satisfies the following axioms:

(1) $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$ $(x \in X)$;

(2) $\|\lambda x\| = |\lambda|\ \|x\|$ $(\lambda \in \mathbb{R},\ x \in X)$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in X)$.

A norm $\|\cdot\|$ is called *decomposable* or *Kantorovich norm* if the following property holds:

(4) for any element $x \in X$ and any decomposition $\|x\| = f_1 + f_2$, $0 \leq f_1, f_2 \in F$, there exist $x_1, x_2 \in X$, such that $x = x_1 + x_2$ and $\|x_k\| = f_k$, $k = 1, 2$.

If decomposition axiom is valid only for disjoint positive elements $f_1, f_2 \in F$, the norm $\|\cdot\|$ is called *disjointly decomposable* or, briefly, *d−-decomposable*.

A pair $(X, \|\cdot\|)$ is called a *lattice-normed space* (shortly, LNS). If the norm $\|\cdot\|$ is decomposable (d-decomposable), then so is the space $(X, \|\cdot\|)$.

A net $(x_\alpha)_{\alpha\in A} \subset X$ *bo*-converges to $x \in X$ if the net $(\|x - x_\alpha\|)_{\alpha\in A}$ *o*-converges to zero in F. A net $(x_\alpha)_{\alpha\in A} \subset X$ is said *bo*-Cauchy if the net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A} \subset X$ is *bo*-converges to zero. An LNS is called *bo*-complete if any *bo*-Cauchy net *bo*-converges. A Banach-Kantorovich space (shortly, BKS) is a d-decomposable *bo*-complete LNS.

If X is a vector lattice then the norm $\|\cdot\|$ is called monotone if $|x| \leq |y|$, $x,\ y \in X$ implies that $\|x\| \leq \|y\|$. BKS with a monotone norm is called a Banach-Kantorovich lattice over F.

Let B be an arbitrary Boolean algebra, $\mathbf{0}$ and $\mathbf{1}$, respectively, its smallest and greatest elements, $a \vee b$ and $a \wedge b$ the exact upper and lower boundaries of the set $\{a, b\}$, $Ca = \mathbf{1} - a$ the complement of a. As usual, the symmetric difference of the elements $a,\ b \in B$ is determined by the equality $a \triangle b = (a \wedge Cb) \vee (Ca \wedge b)$.

A Boolean algebra B is called *complete (σ-complete)* if $\sup A$ and $\inf A$ exist for every (countable) subset $A \subset B$. Let B a complete Boolean algebra. A Boolean subalgebra B_0 is called a regular subalgebra in B if $\sup A,\ \inf A \in B_0$ for any subset $A \subset B_0$.

Elements $a, b \in B$ are called *disjoint,* if $a \wedge b = 0$. A family of elements from B is called *disjoint* if its members are in pairs disjoint. *The decomposition of a unit in Boolean algebra* is an arbitrary set $(e_i)_{i\in I}$ satisfying $\sup\limits_{i\in I} e_i = \mathbf{1}$, $e_i \neq 0$, $e_i \wedge e_j = 0$, $i \neq j$, $i, j \in I$.

Let (Ω, Σ, μ) be a σ-finite measurable space, $L^0 = L^0(\Omega)$ be the algebra of all real measurable functions on (Ω, Σ, μ) (functions equal a.e. are identified). L^0 is an order complete vector lattice with respect to the natural order ($x \leq y \Leftrightarrow y - x \geq 0$ almost everywhere). The weak unit is $\mathbf{1}(\omega) \equiv 1$, and the set $\mathbf{B}(\Omega)$ of all idempotents in L^0 is a complete Boolean algebra.

Let $(X, \|\cdot\|_X)$ be the BKS over L^0. A linear operator $T : X \to L^0$ is called L^0-bounded, if there exists $0 \leq c \in L^0$ such that $|T(x)| \leq c\|x\|_X$ for all $x \in X$. For any L^0-bounded operator T, an element

$$\|T\| := \sup\{|T(x)| : x \in X,\ \|x\|_X \leq \mathbf{1}\} \in L^0,$$

is called the abstract L-norm of the operator T [1, chapter. 4, 4.1.3]. Moreover, the inequality $|T(x)| \le \|T\|\,\|x\|_X$ is always true for each $x \in X$ [1, chapter. 4, 4.1.1].

The set X^* of all L^0-bounded linear mappings from X to L^0 is called the L^0-dual space to the Banach-Kantorovich space X. For $T, S \in X^*$, let $(T+S)(x) = Tx + Sx$, $(\lambda T)(x) = \lambda T x$, where $x \in X$, $\lambda \in \mathbb{R}$. It is clear, that $(T+S), \lambda T \in X^*$ and with respect to the introduced algebraic operations X^* is a linear space over the field $\mathbb{R}$. Moreover, X^*, considered with the abstract norm $\|T\|$, $T \in X^*$, is BKS [1, chapter 4, 4.2.6].

Let B be a complete Boolean algebra. A mapping $m : B \to L^0$ is called an *L^0-valued measure* on B if

1. $m(e) \ge 0$ for any $e \in B$;
2. $m(e \vee g) = m(e) + m(g)$, if $e, g \in B$ and $e \wedge g = 0$;
3. if $e_\alpha \downarrow 0$, $\{e_\alpha\} \subset B$, then $m(e_\alpha) \downarrow 0$.

A measure m is called *strongly positive* if $m(e) = 0$, $e \in B$ implies $e = 0$. A strongly positive measure m is called decomposable if for any $e \in B$ and the decomposition $m(e) = f_1 + f_2$, , $f_1, f_2 \in L^0$ there exist $e_i \in B$ such that $e = e_1 \vee e_2$ and $m(e_i) = f_i$, $i = 1, 2$. It is known that a measure m is decomposable if and only if m is a Maharam measure, i.e. for any $e \in B$, $0 \le f \le m(e)$, $f \in L^0$, there exist $q \in B$, $q \le e$, such that $m(q) = f$ [4].

The following statement shows that, in the case of the Maharam measure $m : B \to L^0$, there is a natural embedding of the Boolean algebra $\mathbf{B}(\Omega)$ into the Boolean algebra B.

Proposition 1 ([4], prop. 3.2). *Let m— L^0-valued Maharam measure on a complete Boolean algebra B. Then there exists a unique Boolean injective completely additive homomorphism $\varphi : \mathbf{B}(\Omega) \to B$, such that*

(i) $\varphi(\mathbf{B}(\Omega))$ is a regular Boolean subalgebra in B.

(ii) m is a φ-modular measure on B, i.e. $m(\varphi(a)e) = a m(e)$ for all $a \in \mathbf{B}(\Omega)$, $e \in B$.

Let $Q(B)$ be the Stone compact of a complete Boolean algebra B, and let $L^0(B) := C_\infty(Q(B))$ be the algebra of all continuous functions $x : Q(B) \to \overline{\mathbb{R}} = [-\infty, +\infty]$ such that $x^{-1}(\{\pm\infty\})$ is a nowhere dense subsets of $Q(B)$. Denote by $C(Q(B))$ the Banach algebra of all continuous real functions on $Q(B)$ with the uniform norm $\|x\|_\infty = \sup\limits_{t \in Q(B)} |x(t)|$. By identifying B with the complete Boolean algebra of all idempotents from $L^0(B)$, we assume that $B \subset L^0(B)$.

Denote by $s(x) = \sup\limits_{n \ge 1}\{|x| > n^{-1}\}$ the support of an element $x \in L^0(B)$, where $\{|x| > \lambda\} \in B$ is the idempotent in the algebra $L^0(B)$, which is the characteristic function χ_{E_λ} of the closure E_λ of the set $\{t \in Q(B) : |x(t)| > \lambda\}$, $\lambda \in \mathbb{R}$.

By to Proposition 1(i), for a L^0-valued Maharam measure m defined on a complete Boolean algebra B, there exists a unique Boolean injective homomorphism

$\varphi : \mathbf{B}(\Omega) \to B$, such that $A = \varphi(\mathbf{B}(\Omega))$ is a regular Boolean subalgebra in B. One can identify $\mathbf{B}(\Omega)$ with a Boolean subalgebra A in B . Moreover, the algebra $L^0(\Omega)$ is identified with the algebra $L^0(A) := C_\infty(Q(A))$. It is also a regular subalgebra in $L^0(B)$, which is also a regular vector sublattice in $L^0(B)$, i.e. sup and inf operations in $L^0(B)$ and $L^0(A)$ coincide.

We denote by t the locally measure topology in L^0 (see, for example, [5, Ch. I, 6.5; Ch.III, 1.3]), and using the L^0—valued metric $\rho(e,g) := m(e \triangle g)$ we define the Hausdorff topology $t(B)$ on the Boolean algebra on B whose basis of neighborhoods around idempotent $e \in B$ is given by

$$\mathrm{V}(e,U) = \{q \in B : \rho(e,q) \in U\},$$

where U is a neighborhood of zero in the topological vector space (L^0, t). It is known that the pair $(B, t(B))$ is a topological Boolean algebra in terminology [6], i.e. the topology $t(B)$ has the following properties:

(1) the operations $e \vee g$, $e \wedge g$ and $Ce = \mathbf{1} - e$ are continuous by variables $e, g \in B$;

(2) if $e_\alpha \downarrow 0$, $(e_\alpha)_{\alpha \in \mathbb{A}} \subset B$, then $e_\alpha \xrightarrow{t(B)} 0$.

Therefore, according to [7, ch.V, § 4], on $L^0(B)$ there is a Hausdorff topology also denoted by $t(B)$ with respect to which $L^0(B)$ is a topological algebra, in particular, the operation of multiplication in $(L^0(B), t(B))$ is continuous in the aggregate of variables; moreover, the cone of positive elements of $L^0(B)_+ = \{x \in L^0(B) : x \geq 0\}$ is closed in $(L^0(B), t(B))$ [7, Ch.V, § 4, Propositions 3, 4]. In addition, the topology $t(B)$ has the following properties:

(T1). For any neighborhood of zero U in $(L^0(B), t(B))$ there exists a neighborhood of zero $V \subset U$, such that from conditions $x \in V$, $a \in C(Q(B))$, $\|a\|_\infty \leq 1$, it follows that $ax \in V$;

(T2). If $\{e_\alpha\}_{\alpha \in \mathbb{A}} \subset B$, and $e_\alpha \xrightarrow{t(B)} 0$, then $x_\alpha e_\alpha \xrightarrow{t(B)} 0$ for any set $\{x_\alpha\}_{\alpha \in \mathbb{A}} \subset L^0(B)$;

(T3). If $e_\alpha \downarrow 0$, $\{e_\alpha\}_{\alpha \in \mathbb{A}} \subset B$, then $e_\alpha \xrightarrow{t(B)} 0$.

A Hausdorff vector topology on $L^0(B)$ satisfying axioms $(T1)-(T3)$ is called the *R-topology* [7, Ch.V, § 4]. It is known that the R-topology is unique on the algebra $L^0(B)$ [7, Ch.V, § 4]..

Let $m : B \to L^0$ be a strongly positive measure on a complete Boolean algebra B, let $\mathcal{S}(B)$ be a vector sublattice in $L^0(B)$, consisting of all B-simple elements of the form $x = \sum\limits_{k=1}^{n} \alpha_k e_k,$, where $\alpha_i \in \mathbb{R}$, $e_i \in B$, $e_i e_j = 0$, $i \neq j$, $i, j = 1, \ldots, n$. The formula

$$I_m(x) := \int x\, dm := \sum_{k=1}^{n} \alpha_k m(e_k),\ x \in \mathcal{S}(B)$$

correctly defines a linear order continuous operator $I_m : \mathcal{S}(B) \to L^0$.

We say that a positive element $x \in L^0(B)$ is *integrable* with respect to m or m-*integrable* if there exists an increasing sequence $(x_n)_{n \in \mathbb{N}}$ of positive elements in $\mathcal{S}(B)$

that (o)-converges to x in $L^0(B)$ and the supremum $\sup\limits_{n\in\mathbb{N}}\int x_n\,dm$ exists in L^0. In this case, the sequence of integrals $(I_m(x_n))_{n\in\mathbb{N}}$ is (o)-fundamental sequence (see [1, 6.1.3]). Therefore, we can determine the integral of the element x by setting

$$I_m(x) := \int x\,dm := \text{(o)-}\lim_{n\to\infty}\int x_n\,dm.$$

An element $x \in L^0(B)$ is *integrable* (= *m-integrable*), if its positive part x_+ and the negative part x_- are integrable. Denote by $L^1(m) := L^1(B,m)$ the set of all integrable elements and for each $x \in L^1(m)$ we put

$$I_m(x) := \int x\,dm := \int x_+\,dm - \int x_-\,dm.$$

It is easy to verify that $L^1(m)$ is an orderly dense ideal in $L^0(B)$ and $I_m : L^1(m) \to L^0$ is a linear operator. For each $x \in L^1(m)$, let $\|x\|_{1,m} := \int |x|\,dm$. Then $(L^1(m), \|x\|_{1,m})$ is a lattice-normed space over L^0 (see [1, 6.1.3]).

Since, according to Proposition 1 (ii)), the Maharam measure on a complete Boolean algebra B is an φ -modular measure on B, this implies the following theorem.

Theorem 1. *([1, Theorem 6.1.10]) Let $m : B \to L^0$ be the Maharam measure on the complete Boolean algebra B. Then*

(i). $(L^1(m), \|x\|_{1,m})$ is a Banach-Kantorovich lattice over L^0;

(ii). $L^0(A)L^1(m) \subset L^1(m)$, morever, $\int \varphi(\alpha)x\,dm = \alpha\int x\,dm$, for all $\alpha \in L^0$, $x \in L^1(m)$, where φ is an isomorphism from $L^0(\Omega)$ into $L^0(A)$; in particular, $|\int \varphi(\alpha)x\,dm| \le |\alpha|\int |x|\,dm$ for all $x \in L^1(m)$, $\alpha \in L^0$.

We need the following well-known properties of Banach-Kantorovich lattice $L^1(m)$ (see, for example, [1]).

Theorem 2. *(i). If $\{x_\alpha\} \subset L^1(m)$, $x_\alpha \xrightarrow{t(B)} x$ and $\{\int |x_\alpha|\,dm\}$ are an orderly bounded net in L^0, then $x \in L^1(m)$ and $\int |x|\,dm \le \sup\limits_\alpha \int |x_\alpha|\,dm$.*

(ii). If $0 \le x_\alpha \in L^1(m)$ for all α, and $x_\alpha \downarrow 0$, then $\|x_\alpha\|_{1,m} \downarrow 0$.

2 Duality for the space $L^1(B,m)$

Let m be an L^0-valued Maharam measure on a complete Boolean algebra B. By Theorem 1, $(L^1(B,m), \|\cdot\|_{1,m})$ is a Banach-Kantorovich lattice over L^0. In particular, the L^0–duality space $(L^1(B,m))^*$ of the space $L^1(B,m)$ is defined. This section provides a complete description of this BKS $(L^1(B,m))^*$.

Let us start with following propositions.

Proposition 2 (cf [1], 5.1.9)**.** *Let $T \in L^1(B,m)^*$, and let φ be an isomorphism from $L^0(\Omega)$ to $L^0(A)$. Then $T(\varphi(z)x) = zT(x)$ for all $z \in L^0(\Omega)$, $x \in L^1(B,m)$.*

Proof. By Theorem 1 (ii) we obtain $\|\varphi(e)x\|_{1,m} = \int \varphi(e)|x|\,dm = e\int |x|\,dm, = \varphi(e)\|x\|_{1,m}$ for any $e \in \mathbf{B}(\Omega),\ x \in L^1(B,m)$. Let $\varphi(e) = a \in A$. Since $T \in L^1(B,m)^*$, we have $|Tx| \leq c\|x\|_{1,m}$ for some $c \in L^0_+(\Omega)$ and any $x \in L^1(B,m)$. Therefore $|T(ax)| \leq ec\|x\|_{1,}$, i.e. support $s(T(ax))$ is majorized by idempotent e. Multiplying the equality of $T(x) = T(ax) + T((\mathbf{1} - a)x)$ by e, we obtain

$$eT(x) = eT(ax) = T(ax).$$

If $z = \sum_{i=1}^n \lambda_i e_i$ is a simple element from $L^0(\Omega)$, where $\lambda_i \in \mathbb{R},\ e_i \in \mathbf{B}(\Omega),\ i = 1,\ldots,n$, then

$$T(\varphi(z)x) = \sum_{i=1}^n \lambda_i T(\varphi(e_i)x) = (\sum_{i=1}^n \lambda_i e_i)T(x) = zT(x).$$

Let z be an arbitrary element from $L^0(\Omega)$ and let $\{z_n\}$ be a sequence of simple elements from $L^0(\Omega)$, such that $z_n \xrightarrow{t} z$. Then $0 \leq |z_n - z| \xrightarrow{t} 0,\ z_n \xrightarrow{t} z$, and, by the inclusion $L^0(A) \subset L^0(B)$, we have that

$$\|\varphi(z_n - z)x\|_{1,m} = \int \varphi(|z_n - z|)|x|\,dm = (|z_n - z|)\|x\|_{1,m} \xrightarrow{t} 0.$$

Since T is continuous, it follows that $z_nT(x) = T(\varphi(z_n)x) \xrightarrow{t} T(\varphi(z)x)$. It remains to use the convergence $z_nT(x) \xrightarrow{t} zT(x)$, due to which $T(\varphi(z)x) = zT(x)$ for all $z \in L^0(\Omega),\ x \in L^1(B,m)$. □

The element $x \in L^0(B)$ is called *A-bounded,* if there exists an element $0 \leq a \in L^0(A)$ such that $|x| \leq a$. Denote by $L^\infty(B,A)$ the set of all A-bounded elements from $L^0(B)$. It is clear that $L^0(A) \subset L^\infty(B,A)$, $C(Q(B)) \subset L^\infty(B,A)$, and $L^\infty(B,A)$ is a subalgebra in $L^0(B)$ with respect to natural algebraic operations in $L^0(B)$.

Proposition 3. *Let $x \in L^0(B)$. The following conditions are equivalent:*

(i) $x \in L^\infty(B,A)$;

(ii) there exists the decomposition of a unit $(\pi_j)_{j\in J} \subset A$ and $(x_j)_{j\in J} \subset C(Q(B))$, such that $x\pi_j = x_j\pi_j$ for all $j \in J$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in L^\infty(B,A)$ and $\pi_0 = \mathbf{1} - s(x),\ \pi_n = E_n(x)E^\perp_{n-1}(x),\ x_n = x\pi_n$, where $E_n(x) = \{|x| > n\}$, $E^\perp_{n-1}(x) = \{|x| \leq n-1\}$. It is clear that $\pi_n\pi_m = 0$ for $n \neq m,\ \sup\limits_{n\geq 0}\pi_n = \mathbf{1},\ |x_n| = |x|\pi_n \leq n\pi_n,\ x\pi_n = x_n\pi_n$,

$(ii) \Rightarrow (i)$. Let $(\pi_j)_{j\in J} \subset A,\ (x_j)_{j\in J}$ — be the corresponding decomposition of a unit and family from $C(Q(B))$ such that $x\pi_j = x_j\pi_j,\ j \in J$. Set $J_n = \{j \in J : n-1 \leq \|x_j\|_\infty < n\},\ p_n = \sup\limits_{j\in J_n}\pi_j$. There is a unique $a_n \in C(Q(B))p_n$, for which $\|a_n\|_\infty \leq n$ and $a_n\pi_j = x_j\pi_j$ for all $j \in J_n$. It is clear $(p_n)_{n=1}^\infty$ is the decomposition of a unit, $p_n \in A,\ xp_n = a_np_n$ and $|x|p_n = |xp_n| = |a_n|p_n \leq np_n$. Take $a \in L^0(A),\ a \geq 0$, such that $ap_n = a_np_n$. Then $|x|p_n \leq ap_n$ and $|x|\sum\limits_{n=1}^k p_n \leq a\sum\limits_{n=1}^k p_n,\ k = 1,2,\ldots$ Since $\sum\limits_{n=1}^k p_n \xrightarrow{t(A)} \mathbf{1}$ as $k \to \infty$, we have $|x| \leq a$, i.e. $x \in L^\infty(B,A)$. □

Remark 1. *The element x from Proposition 3 (ii) is called the mixing of the set $(x_j)_{j\in J}$ with respect to the decomposition of a unit $(\pi_j)_{j\in J}$, and this element is denoted by $mix_{j\in J}\pi_j x_j$ (see [1, 2.2.1]). By Proposition 3 (ii), in $L^\infty(B,A)$ there exists a mixing of any set from $C(Q(B))$ with respect to any decomposition of a unit.*

Remark 2. *The subalgebra $L^\infty(B,A)$ has the following property: if $|x|\le|y|$, $y\in L^\infty(B,A)$, $x\in L^0(B)$, then $x\in L^\infty(B,A)$.*

Indeed, if $x=x_+-x_-\in L^0(B)$, and $(\pi_j)_{j\in J}\subset A$, $(y_j)_{j\in J}$ are the corresponding decomposition of a unit and set from $C(Q(B))$, for which $y\pi_j=y_j\pi_j$, $(y_j)_{j\in J}\subset C(Q(B))$, then $0\le(x_+)\pi_j\le|x|\pi_j\le|y|\pi_j=|y\pi_j|=|y_j|\pi_j$, i.e. $(x_+)\pi_j\in C(Q(B))$. It is similarly shown that $(x_-)\pi_j\in C(Q(B))$. Let $x_j=x\pi_j$. Then $x_j\in C(Q(B))$ and $x\pi_j=x_j\pi_j$ for all $j\in J$, i.e. $x\in L^\infty(B,m)$.

Remark 3. *In the proof of Proposition 3 , it was established that for any $x\in L^\infty(B,A)$ there exists a countable decomposition of a unit $(p_n)_{n=1}^\infty\subset A$, such that $xp_n\in C(Q(B))$ for all $n=1,2,\ldots$*

Let $x\in L^\infty(B,A)$. Set

$$\|x\|_A=\inf\{a\in L^0_+(A):|x|\le a\}.$$

Proposition 4. *The map $\|\cdot\|_A:L^\infty(B,A)\to L^0_+(A)$ is a $L^0(A)$-valued norm on $L^\infty(B,A)$, for which $\|ax\|_A=|a|\|x\|_A$ for all $a\in L^0(A)$, $x\in L^\infty(B,A)$. Moreover, $\|x\|_A=\|\,|x|\,\|_A$ for each $x\in L^\infty(B,A)$, and if $|y|\le|x|$, $y,x\in L^\infty(B,A)$, then $\|y\|_A\le\|x\|_A$.*

Proof. Obviously $\|x\|_A\ge 0$ and $\|0\|_A=0$. Suppose that $x\in L^\infty(B,A)$ and $\|x\|_A=0$. Consider a subset $F=\{a\in L^0_+(A):|x|\le a\}$ of the order-complete vector lattice $L^0(A)$, and for each finite subset $\alpha=\{a_{i_1},\ldots,a_{i_n}\}\subset F$ we put $b_\alpha=\inf\limits_{1\le k\le n}a_{i_k}$. There is a decomposition of a unit $(p_k)_{k=1}^n\subset A$, for which $b_\alpha p_k=a_{i_k}p_k$, $k=1,\ldots,n$. Therefore, $|x|=\sum\limits_{k=1}^n|x|p_k\le\sum\limits_{k=1}^n a_{i_k}p_k=\sum\limits_{k=1}^n b_\alpha p_k=b_\alpha$. We order the set G of all finite subsets of F by inclusion. Then G becomes a direction and $\{b_\alpha\}_{\alpha\in G}\downarrow\inf F=0$. Therefore, $b_\alpha\xrightarrow{t(B)}0$. Since $b_\alpha-|x|\ge 0$ and the cone $L^0_+(B)$ is closed in $(L^0(B),t(B))$, then $-|x|\ge 0$. This means that $x=0$.

Let us show that $\|px\|_A=p\|x\|_A$ for all $p\in A$, $x\in L^\infty(B,A)$. If $a\in L^0_+(B)$ and $|x|\le a$, then $|px|=p|x|\le pa$, and therefore $\|px\|_A\le\inf\{pa:a\in L^0_+(A),\ |x|\le a\}=p\inf\{a\in L^0_+(A):|x|\le a\}=p\|x\|_A$. On the other hand, since $x\in L^\infty(B,A)$, then $|x|\le a$ for some $a\in L^0_+(A)$. Let $b\in L^0_+(A)$ and $|px|\le b$. Then $|x|=p|x|+(\mathbf{1}-p)|x|\le pb+(\mathbf{1}-p)a$, and therefore $\|x\|_A\le pb+(\mathbf{1}-p)a$. Therefore, $p\|x\|_A\le pb\le b$, which implies the inequality $p\|x\|_A\le\|px\|_A$. Thus, $\|px\|_A=p\|x\|_A$.

Now let $0\ne c\in L^0(A)$, $b\in L^0_+(A)$, $b|c|=s(|c|)$. If $a\in L^0_+(A)$, $x\in L^\infty(B,A)$, then $|cx|\le a$ if and only if $s(|c|)|x|=b|cx|\le ba$. Consequently, $\|cx\|_A=\inf\{a\in L^0_+(A):s(|c|)|x|\le b(s(|c|)a)\}=\inf\{a\in L^0_+(A)s(|c|)):s(|c|)|x|\le ba)\}=\inf\{|c|\gamma:\gamma\in L^0_+(A)(s|c|),\ s(|c|)|x|\le\gamma\}=|c|\|s(c)x\|_A=|c|s(|c|)\|x\|_A=|c|\|x\|_A$, in particular, $\|\lambda x\|_A=\|(\lambda\mathbf{1})x\|_A=|\lambda\mathbf{1}|\|x\|_A=|\lambda|\|x\|_A$ for all $\lambda\in\mathbb{R}$, $x\in L^\infty(B,A)$.

Let $a_1, a_2 \in L^0_+(A)$, $x_1, x_2 \in L^\infty(B, A)$ and $|x_1| \le a_1, |x_2| \le a_2$. Then we have $|x_1 + x_2| \le |x_1| + |x_2| \le a_1 + a_2$. This means that $\|x_1 + x_2\|_A \le \|x_1\|_A + \|x_2\|_A$.

The equality $\|x\|_A = \| \, |x| \, \|_A$ and the monotone of the norm follow directly from the definition of the norm. □

Proposition 5. *(i)* $s(\|x\|_A) = \mathbf{1} - \sup\{p \in A : px = 0\}$, *in particular* $s(\|x\|_A)x = x$ *for every* $x \in L^\infty(B, A)$.

(ii) $L^0(A)$*-valued norm* $\| \cdot \|_A$ *is decomposable.*

Proof. (i) If $p \in A$ and $px = 0$, then $p\|x\|_A = \|px\|_A = 0$, i.e. $p \le \mathbf{1} - s(\|x\|_A)$. On the other hand, $\|(\mathbf{1} - s(\|x\|_A)x\| = (\mathbf{1} - s(\|x\|_A))\|x\|_A = 0$. Therefore, $s(\|x\|_A) = \mathbf{1} - \sup\{p \in A) : px = 0\}$, and $x = s(\|x\|_A)x + (\mathbf{1} - s(\|x\|_A))x = s(\|x\|_A)x$.

(ii) Let $a_1, a_2 \in L^0_+(A)$, $x \in L^\infty(B, A)$ and $\|x\|_A = a_1 + a_2$. Take $b \in L^0_+(A)$, for which $b\|x\|_A = s(\|x\|_A)$ and put $x_i = a_i b x$, $i = 1, 2$. Then $x_i \in L^\infty(B, A)$ (see remark 2), $x_1 + x_2 = (a_1 + a_2)bx = s(\|x\|_A)x = x$ (see (i)). Moreover, by of Proposition 3, $\|x_i\|_A = a_i b\|x\|_A = a_i s(\|x\|_A) = a_i$, $i = 1, 2$. □

The following theorem considers a class of L^0-valued L^0-bounded linear mappings defined on $L^1(B, m)$ and generated by elements from $L^\infty(B, A)$.

Theorem 3. *If* $y \in L^\infty(B, A)$, *then* $xy \in L^1(B, m)$ *for all* $x \in L^1(B, m)$ *and the linear map* $T_y(x) = \int xy\, dm$ *from* $L^1(B, m)$ *to* $L^0(\Omega)$ *is* L^0*-bounded. Moreover, if* $\psi = \varphi^{-1}$ *is an isomorphism from* $L^0(A)$ *to* $L^0(\Omega)$, *then*

$$\psi(\|y\|_A) = \sup\{|\int xy\, dm| : x \in L^1(B, m),\ \|x\|_{1,m} \le \mathbf{1}\} = \|T_y\|.$$

Proof. If $y \in L^\infty(B, A)$, then there exists $a \in L_+(A)$, such that $|y| \le a$. The elements $p_0 = \mathbf{1} - s(a)$, $p_n = E_n(a)E^\perp_{n-1}$ — are pair disjoint, belong to A and $\sup\limits_{n \ge 0} p_n = \mathbf{1}$; moreover, $0 \le ap_n \le n$, $n = 1, 2, \ldots$. Consequently, $|yp_n| = |y|p_n \le np_n$, and therefore for all $x \in L^1(B, m)$ we have $x(y\pi_n) \in L^1(B, m)$, where $\pi_n = \sum\limits_{k=0}^{n} p_k$ (see Theorem 1 (ii)). Since $\pi_n \xrightarrow{t(B)} \mathbf{1}$, then $|xy\pi_n| = |xy|\pi_n \xrightarrow{t(B)} |xy|$. By theorem 1 (ii)

$$\int |xy|p_n\, dm \le \int |x|p_n|y|p_n\, dm \le \|\, |y|p_n \,\|_\infty \int |x|p_n\, dm \le n\psi(p_n) \int |x|\, dm.$$

Therefore

$$\int |xy|\pi_n\, dm = \sum_{k=0}^{n} \int |xy|p_k\, dm \le \sum_{k=0}^{n} k\psi(p_k) \int |x|\, dm \le (\psi(a) + \mathbf{1}_{B(\Omega)}) \int |x|\, dm.$$

Hence, by Theorem 2 (i) we have $|xy| \in L^1(B, m)$ and $\|xy\|_{1,m} \le (\psi(a) + \mathbf{1}_{B(\Omega)}) \int |x|\, dm$.

Thus, $xy \in L^1(B, m)$ for all $x \in L^1(B, m)$, and therefore we can define the linear map $T_y : L^1(B, m) \to L^0(\Omega)$, by the equality $T_y(x) = \int xy\, dm$. In addition, we have $|T_y(x)| = |\int xy\, dm| \le \int |xy|\, dm \le (\psi(a) + \mathbf{1})\|x\|_{1,m}$, i.e. the map T_y — L^0-bounded.

We fixed a positive integer k and consider idempotents $p_0^{(k)} = \mathbf{1} - s(a),\ p_n^{(k)} = E_{\frac{n}{k}}(a)E^{\perp}_{\frac{n-1}{k}}(a)$. Repeating the same arguments for $\{p_n^{(k)}\}_{n=0}^{\infty}$ as for the sequence $\{p_n\}_{n=0}^{\infty}$, we obtain the assessment $\|xy\|_{1,m} \leq (\psi(a)+\frac{1}{k}\mathbf{1})\|x\|_{1,m}$. Since k is arbitrary, we have $\|xy\|_{1,m} \leq \psi(a)\|x\|_{1,m}$ for any $a \in L^0_+(A)$, satisfying the inequality $|y| \leq a$. Hence, $|T_y(x)| \leq \psi(\|y\|_A)\|x\|_{1,m}$, i.e. $\|T_y\| \leq \psi(\|y\|_A)$.

Let us now show that $\|T_y\| = \psi(\|y\|_A)$. For $y = 0$ this equality is obvious. If $y \neq 0$ and $\|T_y\| \neq \psi(\|y\|_A)$, then there are $0 \neq e \in B(\Omega),\ e \leq s(\|T_y\|),\ \delta > 0$, such that $\|T_y\|e \leq \psi(\|y\|_A)e - \delta e$.

Let first, $y \geq 0,\ y \neq 0.$. Denote by D the regular Boolean subalgebra in B containing the Boolean algebra A and all idempotents $E_\lambda(y),\ \lambda \geq 0$. Clearly, $L^0(A) \subset L^0(D)$ and $y \in L^0_+(D)$. Set $q = \varphi(e)$ and we fixed $\varepsilon > 0$. Since the inequality $yq \leq \|y\|_A q - \varepsilon q$ — is false, there exists a nonzero idempotent $p \in qD$, such that $yp \geq \|y\|_A p - \varepsilon p$.

Obviously, $p \in B$ and $m(p) \neq 0$. Choose $b \in L^0_+(\Omega)$, for which $bm(p) = s(m(p))$, and put $a = \varphi(b)p$. Because the

$$\|a\|_{1,m} = \int |\varphi(b)p|\, dm = b\int p\, dm = bm(p) = s(m(p)) \leq \mathbf{1},$$

then

$$\|T_y\| \geq |T_y(a)| = \int \varphi(b)py\, dm = b\int py\, dm \geq b\int (\|y\|_A - \varepsilon\mathbf{1})p\, dm =$$

$$b\psi(\|y\|_A - \varepsilon\mathbf{1})m(p) = \psi(\|y\|_A - \varepsilon\mathbf{1})s(m(p))$$

for all $\varepsilon > 0$. Hence, $\|T_y\| \geq \psi(\|y\|_A)s(m(p))$, and therefore

$$\|T_y\|s(m(p)) \geq \psi(\|y\|_A)s(m(p)).$$

Since $p \leq q$, then $s(m(p)) \leq s(m(q)) = s(em(\mathbf{1})) = e$. Therefore, the inequality $\|T_y\|e \leq \psi(\|y\|_A)e - \delta e$ implies the inequality

$$\psi(\|y\|_A)s(m(p)) \leq \|T_y\|s(m(p)) \leq (\psi(\|y\|_A) - \delta)s(m(p)).$$

From the obtained contradiction it follows that $\|T_y\| = \psi(\|y\|_A)$.

If now, $y \leq 0,\ y \neq 0$, then $\|T_y\| = \|T_{-y}\| = \psi(\|(-1)y\|_A) = \psi(\|y\|_A)$. □

Corollary 1. *If $x \in L^1(B,m),\ y \in L^\infty(B,A)$, then $xy \in L^1(B,m)$, in particular, $L^\infty(B,A) \subset L^1(B,m)$. Moreover $|\int xy\, dm| \leq \|x\|_{1,m}\psi(\|y\|_A)$.*

The following theorem describes a L^0-duality space $L^1(B,m)^*$.

Theorem 4. *For any $T \in L^1(B,m)^*$ there exists a unique $y \in L^\infty(B,A)$, such that $T = T_y$.*

Proof. Without loss of generality, one can assume that $m(\mathbf{1}_B) = \mathbf{1}_{B(\Omega)}$.

Let $T \in L^1(B,m)^*$. Choose $a \in L^0_+(\Omega)$, for wich $a\|T\| = s(\|T\|)$. SEt $T_1(x) = aT(x),\ x \in L^1(B,m)$. It is clearly thet $T_1 \in L^1(B,m)^*$ and $\|T_1\| = a\|T\| = s(\|T\|) \leq \mathbf{1}_{B(\Omega)}$. If we show that there exists $y_1 \in L^\infty(B,A)$, for which $T_1 x = \int xy_1\,dm$, then, by Theorem 1 (ii), $Tx = \|T\|T_1(x) = \|T\|\int xy_1\,dm = \int x(\varphi(\|T\|)y_1)\,dm = \int xy\,dm$, where $y = \varphi(\|T\|)y_1 \in L^\infty(B,A)$. Thus, we can also assume that $\|T\| \leq \mathbf{1}_{B(\Omega)}$.

Since $|m(e)| \leq \|e\|_\infty m(\mathbf{1}_B) = \|e\|_\infty \mathbf{1}_{\mathbf{B}(\Omega)}$ is for all $e \in B$, we have $m(B) \subset L^1(\Omega,\Sigma,\mu)$. Since m is a strongly posotove L^0-valued measure on B, then $\nu(e) = \int m(e)\,d\mu,\ e \in B$, is a σ-finite strongly positive numerical measure on B, moreover, $L^1(B,\nu) \subset L^1(B,m)$ and $\|x\|_{1,\nu} = \int(\int |x|\,dm)\,d\nu$ are for all $x \in L^1(B,\nu)$.

Since $|T(x)| \leq \|x\|_{1,m} = \int |x|\,dm \in L^1(\Omega,\Sigma,\mu)$, we have $T(x) \in L^1(\Omega,\Sigma,\mu)$ for any $x \in L^1(B,\nu)$. We define a linear $\mathbb{R}$-valued functional $f(x) = \int T(x)\,d\nu,\ x \in L^1(B,\nu)$ on $L^1(B,\nu)$ Because

$$|f(x)| \leq \int |T(x)|\,d\nu \leq \int \|x\|_{1,m}\,d\nu = \int(\int |x|\,dm)\,d\nu = \|x\|_{1,\nu}$$

for all $x \in L^1(B,\nu)$, then $f \in L^1(B,\nu)^*$. From ([5], Chapter VI, 2.1.) it follows that there exists $y \in L^\infty(B,\nu)$

$$\int T(x)\,d\nu = f(x) = \int xy\,d\nu = \int(\int xy\,dm)\,d\nu$$

for all $x \in L^1(B,\nu)$.

Consider the element $z = T(x) - \int xy\,dm \in L^0(\Omega)$ and $q = s(z_+) - s(z_-)$, $a = \varphi(q)$. Since $a \in A$, we have $ax \in L^1(B,\nu)$ and, By Proposition 2 and Theorem 1 (ii)

$$0 = \int(T(ax) - \int axy\,dm)\,d\nu = \int(q(T(x) - \int xy\,dm))\,d\nu = \int(|T(x) - \int xy\,dm|)\,d\nu,$$

i.e. $T(x) = \int xy\,dm$ for all $x \in L^1(B,\nu)$.

We show that this equality holds for all $x \in L^1(B,m)$. Without loss of generality, one can assume that $x \geq 0$. For elements of $x_n = xE_n(x) \in C(Q(B))_+$ we have that $\|x_n - x\|_{1,m} \xrightarrow{t(B)} 0$ (see Theorem 3). Therefore $T(x_n) \xrightarrow{t(B)} T(x)$ and $|\int x_n y\,dm - \int xy\,dm| \leq \|x_n - x\|_{1,m}\|y\|_\infty \xrightarrow{t(B)} 0$. From this and the equality $T(x_n) = \int x_n y\,dm$ it follows that $T(x) = \int xy\,dm$, i.e. $T = T_y$.

If z — is another element of $L^\infty(B,A)$, for which $T(x) = \int xz\,dm,\ x \in L^1(B,m)$, then $\int x(y-z)\,dm = 0$ for all $x \in L^1(B,m)$. Let $u = y - z$. Taking $x = s(u_+) - s(u_-)$, where $u = y - z$, we obtain $\int |y - z|\,dm = 0$, i.e. $y = z$. □

Since $(L^1(B,m), \|\cdot\|_{1,m})$ is a Banach - Kantorovich space (see Theorem 1), the L^0-dualite space $L^1(B,m)^*$ is (bo)-complete (see [1], 4.2.1). Thus, the following corollary follows from Theorems 4 and 5.

Corollary 2. *A linear space $L^1(B,m)^*$ whith an L^0-valued norm $\|T\|$, $T \in L^1(B,m)^*$, is a Banach - Kantorovich space isometric to $(L^\infty(B,A), \|\cdot\|_A)$.*

Remark 4. *In the case when $A = \mathbf{B}(\Omega)$ is a regular Boolean subalgebra in B and φ — is the identity isomorphism, the statements of Theorems 3 and 4 were obtained in [8].*

3 Normal L^0-bounded linear mappings

Let $B, m, A, L^0 = L^0(\Omega, \Sigma, \mu), \varphi$ — be the same as in Section 2. It follows immediately from Corollary 2 that the map $\|x\|_{\infty,B} = \psi(\|x\|_A)$ defines an L^0-valued norm on $L^\infty(B, A)$, with respect to which $L^\infty(B, A)$ becomes a Banach - Kantorovich space over $L^0(\Omega)$. According to ([1], 4.2.6),the L^0-dualite space $(L^\infty(B, A), \|\cdot\|_{\infty,B})^*$ is also a Banach - Kantorovich space over L^0. In this section, it is shown that $(L^1(B, m), \|\cdot\|_{1,m})$ is isometric to a closed subspace in $(L^\infty(B, A), \|\cdot\|_{\infty,B})^*$, consisting of all normal L^0-bounded linear mappings acting from $L^\infty(B, A)$ to L^0.

A linear L^0-bounded map $T : L^\infty(B, A) \to L^0$ is called *normal* if for any $x_\alpha, x \in L^\infty(B, A)$, $x_\alpha \uparrow x$, implies $T(x_\alpha) \xrightarrow{\mu} T(x)$. The set of all normal linear mappings from $L^\infty(B, A)$ to L^0 is denoted by $L^\infty(B, A)^\sim$. Obviously, $L^\infty(B, A)^\sim$ is a linear subspace in $L^\infty(B, A)^*$.

Proposition 6. *(i) $L^\infty(B, A)^\sim$ — is (bo)-closed in $L^\infty(B, A)^*$;*

(ii) $(L^\infty(B, A)^\sim, \|\cdot\|_{L^\infty(B,A)^*})$ — is the Banach - Kantorovich space over L^0.

Proof. (i) Let $\{T_\alpha\}_{\alpha\in A} \subset L^\infty(B, A)^\sim$, $T \in L^\infty(B, A)^*$ and $\|T_\alpha - T\| \xrightarrow{(o)} 0$. Further, let $\{x_\gamma\}_{\gamma\in\Gamma} \subset L^\infty_+(B, A)$ and $x_\gamma \downarrow 0$. We fixed $\gamma_0 \in \Gamma$ and, for an arbitrary complete neighborhood of zero U in $(L^0(\Omega), t)$ choose a neighborhood of zero V so that $\|x_{\gamma_0}\|_{\infty,B}V + V \subset U$. We fixed $\alpha(V) \in A$, for which $\|T_{\alpha(V)} - T\| \in V$. Since $T_{\alpha(V)} \in L^\infty(B, A)^\sim$, we have $T_{\alpha(V)}(x_\gamma) \xrightarrow{t} 0$, and therefore there exists $\gamma(V) \in \Gamma$, $\gamma(V) \geq \gamma_0$, such that $|T_{\alpha(V)}(x_\gamma)| \in V$ for all $\gamma \geq \gamma(V)$. By of Proposition 4

$$|T(x_\gamma)| \leq |T(x_\gamma) - T_{\alpha(V)}(x_\gamma)| + |T_{\alpha(V)}(x_\gamma)| \leq \|T - T_{\alpha(V)}\| \, \|x_{\gamma_0}\|_{\infty,B} + |T_{\alpha(V)}(x_\gamma)| \in U$$

for all $\gamma \geq \gamma(V)$. Using the fullness of the neighborhood U, we obtain that $T(x_\gamma) \in U$, $\gamma \geq \gamma(V)$, i.e. $Tx_\gamma \xrightarrow{t} 0$.

(ii) By (i), $(L^\infty(B, A)^\sim, \|\cdot\|_{L^\infty(B,A)^*})$ is a (bo)-complete LNS over L^0. It remains to show that the norm $\|\cdot\|_{L^\infty(B,A)^*}$ is decomposable on $L^\infty(B, A)^*$.

Let$T \in L^\infty(B, A)^*$, $b \in L^0$. Set $(bT)(x) = bT(x)$, $x \in L^\infty(B, A)$. Obviously $bT \in L^\infty(B, A)^*$ and $bT \in L^\infty(B, A)^\sim$, in the case when $T \in L^\infty(B, A)^\sim$. Moreover,

$$\|bT\|_{L^\infty(B,A)^*} = \sup\{|bT(x)| : \|x\|_{\infty,B} \leq \mathbf{1}_{\mathbf{B}(\Omega)}\} = |b|\|T\|_{L^\infty(B,A))^*}.$$

As in proposition 5(ii), we obtain that
$\|\cdot\|_{L^\infty(B,A)^*}$ — is a decomposable norm on $L^\infty(B, A)^\sim$. □

Remark 5. *By Theorem 2 (ii) every L^0-bounded linear operator from $L^1(B, m)$ to $L^0(\Omega)$ is normal, and therefore $L^1(B, m)^\sim = L^1(B, m)^*$.*

Remark 6. *Repeating the proof of Proposition 2, we obtain $T(\varphi(z)x) = zT(x)$ for all $T \in L^\infty(B, A)^*$, $z \in L^0(\Omega)$, $x \in L^\infty(B, A)$.*

Theorem 5. *(i) If $y \in L^1(B, m)$, then the linear map $T_y(x) = \int xy\, dm$ from $L^\infty(B, A)$ to $L^0(\Omega)$ is normal. At the same time, $\|T_y\| = \|y\|_{1,m}$.*

(ii) For any $T \in L^\infty(B, A)^\sim$ there is a unique $y \in L^1(B, m)$, for which $T = T_y$.

Proof. (i) Corollary 1 implies that $|T_y(x)| = |\int xy\,dm| \leq \|y\|_{1,m}\|x\|_{\infty,B}$ is for all $x \in L^\infty(B,A)$, which implies the inclusion $T_y \in L^\infty(B,A)^*$ and the assessment $\|T_y\| \leq \|y\|_{1,m}$.

Let $\{x_\alpha\} \subset L^\infty_+(B,A) \subset L^1_+(B,m)$ and $x_\alpha \downarrow 0$. By theorem 2(ii), we have $\|x_\alpha\|_{1,m} \downarrow 0$. Choose a countable decomposition of a unit $\{p_n\}_{n=1}^\infty \subset A$, for which $p_n y \in C(Q(B))$, $n = 1, 2, \dots$ (see Remark 3). Because $\psi(p_n)|\int x_\alpha y\,dm| = |\int x_\alpha p_n y\,dm| \leq \|p_n y\|_\infty \int x_\alpha\,dm \downarrow 0$, then $\psi(p_n)|\int x_\alpha y\,dm| \xrightarrow{t} 0$ for all $n = 1, 2, \dots$, which implies the convergence of $\int x_\alpha y\,dm \xrightarrow{t} 0$. Therefore, $T_y \in L^\infty(B,A)^\sim$. We show now that $\|T_y\| = \|y\|_{1,m}$. For $y = 0$ this equality is obvious. If $y \neq 0$ and $\|T_y\| \neq \|y\|_{1,m}$, then, by the inequality $\|T_y\| \leq \|y\|_{1,m}$, there exist $0 \neq e \in \mathbf{B}(\Omega)$, $e \leq s(\|T_y\|)$, $\varepsilon > 0$, such that $\|T_y\|e \leq \|y\|_{1,m}e - \varepsilon e$. Let $q = s(y_+) - s(y_-) \in B$ and $z = \varphi(e)q$. It is clear that $z \in B$ and $\|z\|_{\infty,B} = \psi(\|z\|_A) = \psi(\varphi(e)\|q\|_A) = e \leq \mathbf{1}_{\mathbf{B}(\Omega)}$. Therefore

$$\|T_y\| \geq |T_y(z)| = \int \varphi(e)qy\,dm = e\int |y|\,dm = e\|y\|_{1,m} \geq \|T_y\|e + \varepsilon e,$$

which is impossible.

The proof of item (ii) is similar to the proof of Theorem 4. □

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