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ERGODIC THEOREMS FOR d-DIMENSIONAL FLOWS IN IDEALS OF COMPACT OPERATORS

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Abstract

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$-algebra of all bounded linear operators acting in $\mathcal{H}$, and let $\mathcal{C}_E$ be the symmetric ideal of compact operators in $\mathcal{H}$ generated by the fully symmetric sequence space $E \subset c_0$. If $T_{\mathbf{u}} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}^d_+$, is a semigroup of positive Dunford-Schwartz operators, which is strongly continuous on $\mathcal{C}_1$, then the following versions of individual and mean ergodic theorems are true: For each $x \in \mathcal{C}_E$ the net $A_t(x) = \frac{1}{t^d} \int\limits_{[0,t]^d} T_{\mathbf{u}}(x) d\mathbf{u}$, $t > 0$, converges to some $\widehat{x} \in \mathcal{C}_E$ with respect to the norm $\|\cdot\|_\infty$, as $t \to \infty$; moreover, if E is separable and $E \neq l_1$ (as a sets), then $\lim\limits_{t\to\infty} \|A_t(x) - \widehat{x}\|_{\mathcal{C}_E} = 0$.

Keywords: *Symmetric sequence space, Banach ideal of compact operators, Dunford-Schwartz operator, individual ergodic theorem, mean ergodic theorem.*

Mathematics Subject Classification (2010): *46E30, 37A30, 47A35.*

Introduction

Let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$–algebra of all bounded linear operators in a complex infinite-dimensional Hilbert space $\mathcal{H}$. The study of noncommutative individual ergodic theorems in the space of measurable operators affiliated with a semifinite von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ equipped with a faithful normal semifinite trace τ was initiated by F. Yeadon [16], who was established that for any positive $L^1 - \mathcal{M}$ – contraction $T : L^1 \to L^1$ and every $x \in L^1$ there exists $\widehat{x} \in L^1$ such that the averages $A_n(T)(x) = \frac{1}{n+1} \sum\limits_{k=0}^{n} T^k(x)$ converge to $\widehat{x}$ bilaterally almost uniformly, that is, given $\varepsilon > 0$, there exists a projection $e \in \mathcal{M}$ such that $\tau(\mathbf{1} - e) < \varepsilon$ and $\|e(A_n(T)(x) - \widehat{x})e\|_\infty \to 0$ as $n \to \infty$, where $\mathbf{1}$ is the unit of $\mathcal{M}$.

The study of individual ergodic theorems beyond $L^1(\mathcal{M}, \tau)$ started much later with another fundamental paper by M. Junge and Q. Xu [11], where, among other results, individual ergodic theorem was extended to the case with a positive Dunford-Schwartz operator acting in the space $L^p(\mathcal{M}, \tau)$, $1 < p < \infty$. In [2] ([3]) an individual ergodic theorem was proved for a positive Dunford-Schwartz operator in a noncommutative Lorentz (respectively, Orlicz) space.

Advancing Lance's extension of the pointwise ergodic theorem for actions of the group of integers on von Neumann algebras, Conze and Dang-Ngoc [4] and Watanabe [15] studied continuous extensions of Lance's results. In particular, the noncommutative individual ergodic theorems were established for actions of the semigroups $\mathbb{R}^d_+$. The corresponding ergodic theorem for actions of $\mathbb{R}^d_+$ and with respect to bilaterally almost uniform convergence was initially considered by Junge and Xu [11]. In particular, they derived that these averages converge bilaterally almost uniformly in any noncommutative L^p-space for $1 \leq p < \infty$ and almost uniformly if $2 < p < \infty$.

Let $\mathcal{H}$ be a complex infinite-dimensional Hilbert space, and let c_0 be the Banach space of converging to zero sequences of complex numbers. Every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ generates a symmetric ideal of compact operators $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$, acting in $\mathcal{H}$, by the following rule (see, for example, [13, Chapter 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E,$$

where $\mathcal{K}(\mathcal{H})$ is the two-sided ideal of compact linear operators in $\mathcal{B}(\mathcal{H})$ and $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of eigenvalues of the compact operator $|x|$ in the decreasing order.

Let $\mathbb{N}$ (respectively, $\mathbb{R}$) be the set of natural (respectively, real) numbers. Fix $d \in \mathbb{N}$ and denote $\mathbb{R}_+^d = \{\mathbf{u} = (u_1, \ldots, u_d) : 0 \leq u_i \in \mathbb{R}, \ i = 1, \ldots, d\}$. Let $\{T_\mathbf{u}\}_{\mathbf{u}\in\mathbb{R}_+^d}$ be a semigroup of positive Dunford-Schwartz operators, such that $T_\mathbf{0}(x) = x$ for all $x \in \mathcal{B}(\mathcal{H})$ (definition of Dunford-schwartz operator see below in Preliminaries section). A semigroup $\{T_\mathbf{u}\}_{\mathbf{u}\in\mathbb{R}_+^d}$ is said to be *strongly continuous* on $\mathcal{C}_{l_1}$ if

$$\lim_{\mathbf{u}\to\mathbf{v}} \|T_\mathbf{u}(x) - T_\mathbf{v}(x)\|_{\mathcal{C}_{l_1}} = 0.$$

for each $x \in \mathcal{C}_{l_1}$.

Let $\{T_\mathbf{u}\}_{\mathbf{u}\in\mathbb{R}_+^d}$ be a strongly continuous on $\mathcal{C}_{l_1}$ semigroup of Dunford-Schwartz operators and let

$$A_t(x) = \frac{1}{t^d} \int_{[0,t]^d} T_\mathbf{u}(x) d\mathbf{u}, t > 0, \tag{1}$$

be the corresponding ergodic averages. We show that the operators A_t extend to the Dunford-Schwarz operators, in particular, $A_t(\mathcal{C}_E) \subset \mathcal{C}_E$ for any fully symmetric sequence space $E \subset c_0$. Therefore, we can talk about the convergence of the averages $A_t(x)$ as $t \to \infty$.

We prove the following noncommutative version of the individual ergodic theorem for d-dimensional flows in $\mathcal{C}_E$: For each $x \in \mathcal{C}_E$ the corresponding ergodic averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_E$ with respect to the uniform norm $\|\cdot\|_\infty$ as $t \to \infty$. Besides, we prove the following noncommutative version of the mean ergodic theorem for d-dimensional flows in $\mathcal{C}_E$: If $(E, \|\cdot\|_E)$ is separable space and $\mathcal{C}_E \neq \mathcal{C}_{l_1}$ as sets, then averages $A_t(x)$ converge to $\widehat{x}$ with respect to the norm $\|\cdot\|_{\mathcal{C}_E}$ as $t \to \infty$. In conclusion, we give an illustration of the obtained ergodic theorems for the Orlicz, Lorentz, and Marcinkiewicz ideals of compact operators.

1 Preliminaries

Let l^∞ (respectively, c_0) be the Banach space of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm

$\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$. If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the non-increasing rearrangement $\xi^* = \{\xi_n^*\}_{n=1}^\infty$ of ξ is defined by

$$\xi_n^* = \inf\left\{\sup_{n\notin F} |\xi_n| : F \subset \mathbb{N},\ |F| < n\right\}.$$

The Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^{m} \xi_n^* \leq \sum_{n=1}^{m} \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E,\ \xi \in l^\infty,\ \xi^* \leq \eta^* \text{ (respectivelly, } \xi^* \prec\prec \eta^*) \implies \xi \in E \text{ and } \|\xi\|_E \leq \|\eta\|_E.$$

Let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional separable Hilbert space over the field $\mathbb{C}$ of complex numbers, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the C^*-algebra of bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ the two-sided ideal of compact linear operators in $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \geq 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0,\infty]$ be the canonical trace on $\mathcal{B}(\mathcal{H})$, that is, $\tau(x) = \sum\limits_{i=1}^{\infty}(x\varphi_i, \varphi_i)$ for all $x \in \mathcal{B}_+(\mathcal{H})$, where $\{\varphi_i\}_{i=1}^\infty$ is an orthonormal basis in $\mathcal{H}$.

Let $\mathcal{P}(\mathcal{H})$ be the lattice of projections in $\mathcal{H}$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda\}_{\lambda\geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of x, that is, $e_\lambda = \{|x| \leq \lambda\}$. If $t > 0$, then the t-th generalized singular number of x, or the non-increasing rearrangement of x, is defined as (see [9])

$$\mu_t(x) = \inf\{\lambda > 0 :\ \tau(e_\lambda^\perp) \leq t\}.$$

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called noncommutative symmetric (fully symmetric) space if the conditions

$$x \in X,\ y \in \mathcal{B}(\mathcal{H}),\ \mu_t(y) \leq \mu_t(x)\ \forall\ t > 0$$

$$\text{(respectively, } \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt\ \forall\ s > 0 \text{ (writing } y \prec\prec x))$$

imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$, as well as the classical Banach two-sided ideals

$$\mathcal{C}_p = \{x \in \mathcal{K}(\mathcal{H}) :\ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\},\ 1 \leq p < \infty,$$

are examples of noncommutative fully symmetric spaces.

If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum\limits_{n=1}^{m(x)} s_n(x)p_n$ (if $m(x) = \infty$, the series converges with respect to the uniform norm $\|\cdot\|_\infty$), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^\infty$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Fix an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C}\cdot\varphi_n \subset \mathcal{H}$. If $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ is a symmetric space then the set

$$E(X) = \left\{\xi = \{\xi_n\}_{n=1}^\infty \in c_0 : \ x_\xi = \sum_{n=1}^\infty \xi_n p_n \in X\right\}$$

(the series converges uniformly) is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$. Consequently, each symmetric space $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [14, Chapter 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

The pair $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is called a Banach ideal of compact operators (cf. [10, Chapter III]). It is known that $(\mathcal{C}_p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$ for all $1 \le p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$.

Hardy-Littlewood-Polya partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec\prec y, \ x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

Using Lemma 2.5 (*ii*) [9] and Theorem 4.4 (*iii*) [9] we get

$$x \prec\prec y, z \prec\prec y, \ x, y, z \in \mathcal{K}(\mathcal{H}) \Longrightarrow x + z \prec\prec 2y. \tag{1}$$

We say that a Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is fully symmetric, if conditions $y \in \mathcal{C}_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec\prec y$ entail that $x \in \mathcal{C}_E$ and $\|x\|_{\mathcal{C}_E} \le \|y\|_{\mathcal{C}_E}$. It is clear that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

Note that, along with any Schatten ideals $\mathcal{C}_p$, $1 \le p < \infty$, of compact operators, the family of such fully symmetric ideals $\mathcal{C}_E$ contains many noncommutative counterparts of classical symmetric sequence spaces, examples of which are given in the last section of this note.

A linear contraction $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called a Dunford-Schwartz operator (writing $T \in DS$), if $T(\mathcal{C}_1) \subset \mathcal{C}_1$ and $\|T(x)\|_1 \le \|x\|_1$ for all $x \in \mathcal{C}_1$. We will write $T \in DS^+$ if T is a positive Dunford-Schwartz operator, that is, $T \in DS$ and $T(\mathcal{B}_+(\mathcal{H})) \subset \mathcal{B}_+(\mathcal{H})$.

Any fully symmetric ideal $\mathcal{C}_E$ is an exact interpolation space in the Banach pair $(\mathcal{C}_1, \mathcal{B}(\mathcal{H}))$ (see [5, Theorem 2.4]). It then follows that $T(\mathcal{C}_E) \subset \mathcal{C}_E$ and

$\|T\|_{\mathcal{C}_E \to \mathcal{C}_E} \leq 1$ for all $T \in DS$. In particular, $T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$ and the restriction of T on $\mathcal{K}(\mathcal{H})$ is a linear contraction (also denoted by T). In addition, $T(x) \prec\prec x$ for all $T \in DS$ and $x \in \mathcal{K}(\mathcal{H})$ (see [6, Theorem 4.7]).

In [1] established the following result about an extension of any positive linear contraction $T : \mathcal{C}_1 \to \mathcal{C}_1$ with the property $\|T(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$ up to the Dunford-Schwarz operator $\widetilde{T} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$.

Theorem 1. *Let $T : \mathcal{C}_1 \to \mathcal{C}_1$ be a positive linear contraction such that $\|T(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$. Then there exists a unique operator $\widetilde{T} \in DS$ such that $\widetilde{T}(x) = T(x)$ for all $x \in \mathcal{C}_1$, and $\widetilde{T}$ is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ - continuous.*

2 Individual and Mean Ergodic theorems for d-dimensional flows in Banach ideals of compact operators

Let $d \in \mathbb{N}$ and let $\mathbb{R}^d_+ = \{\mathbf{u} = (u_1, \ldots, u_d) : \ 0 \leq u_i \in \mathbb{R}, \ i = 1, \ldots, d\}$. In what follows, $\{T_\mathbf{u}\}_{\mathbf{u} \in \mathbb{R}^d_+} \subset DS^+$ is a semigroup such that $T_0(x) = x$ for all $x \in \mathcal{B}(\mathcal{H})$. A semigroup $\{T_\mathbf{u}\}_{\mathbf{u} \in \mathbb{R}^d_+}$ is said to be strongly continuous on fully symmetric ideal $\mathcal{C}_1$, if $\lim\limits_{\mathbf{u} \to \mathbf{v}} \|T_\mathbf{u}(x) - T_\mathbf{v}(x)\|_{\mathcal{C}_1} = 0$ for each $x \in \mathcal{C}_1$. Note that for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d_+$, $\mathbf{u} = (u_1, \ldots, u_d)$, $\mathbf{v} = (v_1, \ldots, v_d)$ the convergence $\mathbf{u} \to \mathbf{v}$ means that $u_i \to v_i$ for each $i = 1, \ldots, d$.

If $\{T_\mathbf{u}\}_{\mathbf{u} \in \mathbb{R}^d_+} \subset DS^+$ is a strongly continuous semigroup on $\mathcal{C}_1$ then for any $x \in \mathcal{C}_1$ and $y \in \mathcal{B}(\mathcal{H})$ the function $\varphi_{x,y}(\mathbf{u}) = \tau(T_\mathbf{u}(x)y)$ is continuous on $\mathbb{R}^d_+$. Therefore, for the Lebesgue measure μ on $\mathbb{R}^d_+$ we have that the map $U_x : \mathbb{R}^d_+ \to \mathcal{C}_1$ defined as $U_x(\mathbf{u}) = T_\mathbf{u}(x)$ is weakly μ-measurable, that is, the complex function $\tau(U_x(\mathbf{u})y)$ is a measurable function on $(\mathbb{R}^d_+, \mu)$ for all $y \in \mathcal{B}(\mathcal{H})$ (recall that $(\mathcal{C}_1)^* = \mathcal{B}(\mathcal{H})$ and every $f \in (\mathcal{C}_1)^*$ has the following form $f(x) = \tau(xy)$ for some $y \in \mathcal{B}(\mathcal{H})$). Since, in addition, $U_x(\mathbb{R}^d_+)$ is a separable subset in $\mathcal{C}_1$, Pettis theorem [17, Chapter V, § 4] entails that the real function $\|U_x(\mathbf{u})\|_1 = \|T_\mathbf{u}(x)\|_1$ is μ-measurable on $\mathbb{R}^d_+$. Using the inequality $\|T_\mathbf{u}(x)\|_1 \leq \|x\|_1$, we obtain that $\|T_\mathbf{v}(x)\|_1$ is an integrable function on $[0, u_1] \times [0, u_2] \times \ldots \times [0, u_d]$ for any $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d_+$. By [17, Chapter V, § 5, Theorem 1], the function $T_\mathbf{v}(x)$ is Bochner μ-integrable on $[0, t]^d$ for every $t > 0$. Consequently, for any $x \in \mathcal{C}_1$ and $t > 0$ there exists the Bochner integral $A_t(x) = \frac{1}{t^d} \int_{[0,t]^d} T_\mathbf{u}(x) d\mathbf{u} \in \mathcal{C}_1$. It is clear that $\|A_t(x)\|_1 \leq \|x\|_1$ and $\|A_t(x)\|_\infty \leq \|x\|_\infty$ for all $x \in \mathcal{C}_1$. Consequently, by Theorem 1, there exists a unique operator $\widetilde{A_t} \in DS^+$ such that $\widetilde{A_t}(x) = A_t(x)$ for all $x \in \mathcal{C}_1$, and $\widetilde{A_t}$ is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ - continuous. Below, the operator $\widetilde{A_t}$ is denoted by A_t.

Theorem 2. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric Banach ideal, and let $\{T_\mathbf{u}\}_{\mathbf{u} \in \mathbb{R}^d_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then given $x \in \mathcal{C}_E$, the averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_E$ with respect to the uniform norm $\|\cdot\|_\infty$ as $t \to \infty$.*

Proof. Using Theorem 6.6 [11], we get that for each $x \in \mathcal{C}_2$ there exists $\widehat{x} \in \mathcal{C}_2$ such that the averages $A_t(x)$ converge to $\widehat{x} \in \mathcal{C}_2$ bilaterally almost uniformly, that is, given $\varepsilon > 0$, there exists a projection $e \in \mathcal{P}(\mathcal{H})$ such that $\tau(\mathbf{1}-e) < \varepsilon$ and $\|e(A_t(x)-\widehat{x})e\|_\infty \to 0$ as $t \to \infty$, where τ is the canonical trace on $\mathcal{B}(\mathcal{H})$. In particular, for $\varepsilon = \frac{1}{2}$ we obtain that $\mathbf{1}-e=0$, thus $\|A_t(x)-\widehat{x}\|_\infty \to 0$ as $t \to \infty$. Consequently,

$$\|A_t(x) - A_s(x)\|_\infty \to 0 \ \text{ as } \ t,s \to \infty \ \ \forall \ \ x \in \mathcal{C}_2. \tag{2}$$

Since $\mathcal{C}_2$ contains the linear subspace of all finite-dimensional operators, it follows that $\mathcal{C}_2$ is everywhere dense in $\mathcal{K}(\mathcal{H})$. Therefore, for each $x \in \mathcal{C}_E$ and $\varepsilon > 0$, there exists $x_\varepsilon \in \mathcal{C}_2$ such that $\|A_t(x) - A_t(x_\varepsilon)\|_\infty \le \|x - x_\varepsilon\|_\infty < \varepsilon$ for all $t > 0$. Using (2) and following inequalities

$$\|A_t(x)-A_s(x)\|_\infty \le \|A_t(x)-A_t(x_\varepsilon)\|_\infty + \|A_t(x_\varepsilon)-A_s(x_\varepsilon)\|_\infty + \|A_s(x_\varepsilon)-A_s(x)\|_\infty =$$

$$= \|A_t(x-x_\varepsilon)\|_\infty + \|A_t(x_\varepsilon)-A_s(x_\varepsilon)\|_\infty + \|A_s(x_\varepsilon - x)\|_\infty \le 2\varepsilon + \|A_t(x_\varepsilon)-A_s(x_\varepsilon)\|_\infty$$

we obtain that

$$\|A_t(x) - A_s(x)\|_\infty \to 0 \ \text{ as } \ t,s \to \infty.$$

Since $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ is the Banach space it follows that there exists $\widehat{x} \in \mathcal{K}(\mathcal{H})$ such that $\|A_t(x) - \widehat{x}\|_\infty \to 0$ as $t \to \infty$.

By [10, Chapter II, §2, Corollary 2.3] we have that $|s_n(A_t(x)) - s_n(\widehat{x})| \le \|A_t(x) - \widehat{x}\|_\infty$ for all $n \in \mathbb{N}$. Consequently, $s_n(A_t(x)) \to s_n(\widehat{x})$ as $t \to \infty$ for all $n \in \mathbb{N}$. Since $A_t \in DS^+$ it follows that $A_t(x) \prec\prec x$ (see [6, Theorem 4.7]), that is

$$\sum_{n=1}^{m} s_n(A_t(x)) \le \sum_{n=1}^{m} s_n(x) \quad \text{for all} \quad m \in \mathbb{N} \ \text{ and } \ t > 0.$$

Thus

$$\sum_{n=1}^{m} s_n(\widehat{x}) \le \sum_{n=1}^{m} s_n(x) \quad \text{for all} \quad m \in \mathbb{N},$$

that is $\widehat{x} \prec\prec x \in \mathcal{C}_E$. Finally, using that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric Banach ideal we obtain that $\widehat{x} \in \mathcal{C}_E$. □

Remark. An analogue of Theorem 2 for the ergodic averages $A_t(x) = \frac{1}{t}\int\limits_0^t T_s(x)ds$, $t > 0$, of strongly continuous on $\mathcal{C}_1$ semigroup of Dunford-Schwartz operators $\{T_t\}_{t\ge 0}$ was obtained in [1], where it was proved that for any fully symmetric Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ for each $x \in \mathcal{C}_E$ the averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_E$ (as $n \to \infty$) with respect to the uniform norm $\|\cdot\|_\infty$ as $t \to \infty$.

Using the reflexivity of $\mathcal{C}_p$-spaces, $1 < p < \infty$, and the well-known mean ergodic theorem for linear contractions of reflexive spaces (see, for example, [8, Chapter VII, §5, Corollary 4]), we have the following version of the mean ergodic theorem for the Dunford-Schwarz operators:

If $T \in DS$ and $1 < p < \infty$, then the averages $A_n(T) = \frac{1}{n}\sum\limits_{k=0}^{n-1} T^k$ converge strongly in $\mathcal{C}_p$, that is, given $x \in \mathcal{C}_p$, there exists $\widehat{x} \in \mathcal{C}_p$ such that $\|A_n(T)(x) - \widehat{x}\|_p \to 0$ as

$n \to \infty$. If $p = 1$, this is not true in general. As a consequence, mean ergodic theorem may not hold in some fully symmetric Banach ideals $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$.

It is known that every separable symmetric sequence space $(E, \|\cdot\|_E)$ is a fully symmetric sequence space. In this case a symmetric Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal. In [1] we proved the following version of the mean ergodic theorem for the ergodic averages $A_t(x) = \frac{1}{t}\int\limits_0^t T_s(x)ds$, $t > 0$, of strongly continuous on $\mathcal{C}_1$ semigroup of Dunford-Schwartz operators $\{T_t\}_{t\geq 0}$.

Theorem 3. *Let $(E, \|\cdot\|_E)$ be a separable symmetric sequence space and $E \neq l^1$ as sets. Then for any strongly continuous semigroup $\{T_t\}_{t\in\mathbb{R}_+} \subset DS^+$ on $\mathcal{C}_1$ the averages A_t converge strongly in $\mathcal{C}_E$.*

It is clear that Theorem 3 cannot be used for the ideal $(\mathcal{C}_1, \|\cdot\|_1)$.

The following theorem essentially extends result of Theorem 3 for action of the group $\mathbb{R}_+^d$ for every $d \in \mathbb{N}$.

Theorem 4. *Let $(E, \|\cdot\|_E)$ be a separable symmetric sequence space and $E \neq l^1$ as sets. Then for any strongly continuous semigroup $\{T_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{R}_+^d} \subset DS^+$ on $\mathcal{C}_1$ the averages A_t converge strongly in $\mathcal{C}_E$.*

To prove the Theorem 4, we need the following property of separable symmetric sequence spaces [7, Proposition 2.2].

Proposition 1. *Let $(E, \|\cdot\|_E)$ be a separable symmetric sequence space and $E \neq l^1$ as sets. If $y_n \in \mathcal{C}_E$, $y_n \prec\prec x \in \mathcal{C}_E$ for every $n \in \mathbb{N}$ and $\|y_n\|_\infty \to 0$ as $n \to \infty$, then $\|y_n\|_{\mathcal{C}_E} \to 0$ as $n \to \infty$.*

Proof of Theorem 4. By Theorem 2, for every $x \in \mathcal{C}_E$ there exists $\widehat{x} \in \mathcal{C}_E$ such that $\|A_t(x) - \widehat{x}\|_\infty \to 0$ as $t \to \infty$. Moreover, $(-\widehat{x}) \prec\prec \widehat{x} \prec\prec x$ and $A_t(x) \prec\prec x$ for every $t > 0$ (see proof of Theorem 2). Using (1) we obtain that $A_t(x) + (-\widehat{x}) \prec\prec 2x \in \mathcal{C}_E$. Consequently, by Proposition 1, for every sequence $0 < t_n \to \infty$ we have that $\|A_{t_n}(x) - \widehat{x}\|_{\mathcal{C}_E} \to 0$ as $n \to \infty$. It means that $\|A_t(x) - \widehat{x}\|_{\mathcal{C}_E} \to 0$ as $t \to \infty$. $\square$

3 Applications to Orlicz, Lorentz and Marcinkiewicz Banach ideals

In this section we give applications of Theorems 2 and 4 to Orlicz, Lorentz, and Marcinkiewicz ideals of compact operators.

1. Let Φ be an *Orlicz function*, that is, $\Phi : [0, \infty) \to [0, \infty)$ is left-continuous, convex, increasing and such that $\Phi(0) = 0$ and $\Phi(u) > 0$ for some $u \neq 0$. Let

$$l_\Phi(\mathbb{N}) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \sum_{n=1}^\infty \Phi\left(\frac{|\xi_n|}{a}\right) < \infty \text{ for some } a > 0 \right\}$$

be the corresponding *Orlicz sequence space*, and let $\|\xi\|_\Phi = \inf\left\{a>0: \sum\limits_{n=1}^{\infty}\Phi\left(\frac{|\xi_n|}{a}\right)\le 1\right\}$ be the *Luxemburg norm* in $l_\Phi(\mathbb{N})$. It is well-known that $(l_\Phi(\mathbb{N}), \|\cdot\|_\Phi)$ is a fully symmetric sequence space.

If $\Phi(u) > 0$ for all $u \neq 0$, then $\sum\limits_{n=1}^{\infty}\Phi\left(\frac{1}{a}\right) = \infty$ for each $a > 0$. Hence $\mathbf{1} = \{1, 1, ...\} \notin l_\Phi(\mathbb{N})$ and $l_\Phi(\mathbb{N}) \subset c_0$. If $\Phi(u) = 0$ for all $0 \le u < u_0$, then $\mathbf{1} \in l_\Phi$ and $l_\Phi(\mathbb{N}) = l^\infty$.

It is said that an Orlicz function Φ satisfies (Δ_2)-*condition at* 0 if there exist $u_0 \in (0,\infty)$ and $k > 0$ such that $\Phi(2u) < k\cdot\Phi(u)$ for all $0 < u < u_0$. It is well known that an Orlicz function Φ satisfies (Δ_2)-condition at 0 if and only if $(l_\Phi(\mathbb{N}), \|\cdot\|_\Phi)$ is a separable space.

We also note that $l_\Phi(\mathbb{N}) = l^1$ as sets if and only if $\limsup\limits_{u\to 0}\frac{\Phi(u)}{u} > 0$ [14, Chapter 16, §16.2].

Set $\mathcal{C}_\Phi = \mathcal{C}_{l_\Phi(\mathbb{N})}$ and $\|x\|_\Phi = \|x\|_{\mathcal{C}_{l_\Phi(\mathbb{N})}}$, $x \in \mathcal{C}_\Phi$. Theorems 2 and 4 are yield the following.

Theorem 5. *Let Φ be an Orlicz function, $\Phi(u) > 0$ for all $u > 0$, $x \in \mathcal{C}_\Phi$ and let $\{T_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{R}^d_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then*

(i). The averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_\Phi$ with respect to the uniform norm as $t \to \infty$;

(ii). If $\lim\limits_{u\to 0}\frac{\Phi(u)}{u} = 0$ and the Orlicz function Φ satisfy (Δ_2)-condition at 0, then $\|A_t(x) - \widehat{x}\|_\Phi \to 0$ as $t \to \infty$.

2. Let ψ be a concave function on $[0,\infty)$ with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$, and let

$$\Lambda_\psi(\mathbb{N}) = \left\{\xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty : \ \|\xi\|_{\Lambda_\psi} = \sum_{n=1}^{\infty}\xi_n^*(\psi(n) - \psi(n-1)) < \infty\right\}$$

be the corresponding *Lorentz sequence space*. It is well-known that $(\Lambda_\psi(\mathbb{N}), \|\cdot\|_{\Lambda_\psi})$ is a fully symmetric sequence space (see, for example, [14, Part III, Ch.9, § 9.1]); moreover, if $\psi(\infty) = \infty$, then $\mathbf{1} \notin \Lambda_\psi(\mathbb{N})$ and $\Lambda_\psi(\mathbb{N}) \subset c_0$. If $\psi(\infty) < \infty$, then $\mathbf{1} \in \Lambda_\psi(\mathbb{N})$ and $\Lambda_\psi(\mathbb{N}) = l^\infty$. In addition, the space $(\Lambda_\psi(\mathbb{N}), \|\cdot\|_{\Lambda_\psi})$ is separable if and only if $\psi(+0) = 0$ and $\psi(\infty) = \infty$ [14, Ch.9, §9.3, Theorem 9.3.1]. It is clear that $\lim\limits_{t\to\infty}\frac{\psi(t)}{t} > 0$ if and only if the norms $\|\cdot\|_{\Lambda_\psi}$ and $\|\cdot\|_1$ are equivalent on $\Lambda_\psi(\mathbb{N})$, i.e. the equality $\Lambda_\psi(\mathbb{N}) = l^1$ (as sets) is true.

Set $\mathcal{C}_{\Lambda_\psi} = \mathcal{C}_{\Lambda_\psi(\mathbb{N})}$ and $\|x\|_{\Lambda_\psi} = \|x\|_{\mathcal{C}_{\Lambda_\psi(\mathbb{N})}}$, $x \in \mathcal{C}_{\Lambda_\psi}$. Theorems 2 and 4 are imply the following.

Theorem 6. *Let ψ be a concave function on $[0,\infty)$ with $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$, let $x \in \mathcal{C}_{\Lambda_\psi}$ and let $\{T_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{R}^d_+} \subset DS^+$ be a strongly continuous semigroup on $\mathcal{C}_1$. Then*

(i). If $\psi(\infty) = \infty$, then the averages $A_t(x)$ converge to some $\widehat{x} \in \mathcal{C}_{\Lambda_\psi}$ with respect to the uniform norm;

(ii). If $\psi(+0) = 0,\ \psi(\infty) = \infty$ *and* $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$, *then* $\|A_t(x) - \widehat{x}\|_{\Lambda_\psi} \to 0$ *as* $t \to \infty$.

3. Let ψ be as above, and let

$$M_\psi(\mathbb{N}) = \left\{\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \ \|\xi\|_{M_\psi} = \sup_{n\in\mathbb{N}} \frac{1}{\psi(n)} \sum_{i=1}^n \xi_i^* \cdot (\psi(i) - \psi(i-1)) < \infty\right\}$$

be the corresponding *Marcinkiewicz sequence space*. It is known that $(M_\psi(\mathbb{N}), \|\cdot\|_{M_\psi})$ is a fully symmetric space. Moreover, $\mathbf{1} \notin M_\psi(\mathbb{N})$ if and only if $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$ [12, Ch. II, § 5]. Thus, in this case we have that $M_\psi(\mathbb{N}) \subset c_0$. Set $\mathcal{C}_{M_\psi} = \mathcal{C}_{M_\psi(\mathbb{N})}$ and $\|x\|_{M_\psi} = \|x\|_{\mathcal{C}_{M_\psi(\mathbb{N})}}$, $x \in \mathcal{C}_{M_\psi}$. Theorem 2 is imply the following.

Theorem 7. *Let* ψ *be a concave function on* $[0,\infty)$ *with* $\psi(0) = 0,\ \psi(t) > 0$ *for all* $t > 0$, *and* $\lim\limits_{t\to\infty} \frac{\psi(t)}{t} = 0$. *Let* $x \in \mathcal{C}_{M_\psi}$ *and let* $\{T_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{R}_+^d} \subset DS^+$ *be a strongly continuous semigroup on* $\mathcal{C}_1$. *Then for any* $x \in \mathcal{C}_{M_\psi}$ *the averages* $A_t(x)$ *converge to some* $\widehat{x} \in \mathcal{C}_{M_\psi}$ *with respect to the uniform norm.*

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