

3-15-2021

Nonlocal problems for a fractional order mixed parabolic equation

Azizbek Mamanazarov

Fergana State University, mamanazarovaz1992@gmail.com

Follow this and additional works at: <https://bulletin.nuu.uz/journal>



Part of the [Partial Differential Equations Commons](#)

Recommended Citation

Mamanazarov, Azizbek (2021) "Nonlocal problems for a fractional order mixed parabolic equation," *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*: Vol. 4: Iss. 1, Article 1. DOI: <https://doi.org/10.56017/2181-1318.1143>

This Article is brought to you for free and open access by Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences. It has been accepted for inclusion in Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences by an authorized editor of Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences. For more information, please contact karimovja@mail.ru.

NONLOCAL PROBLEMS FOR A FRACTIONAL ORDER MIXED PARABOLIC EQUATION

MAMANAZAROV A.O.

Fergana State University, Fergana, Uzbekistan

e-mail: mamanazarovaz1992@gmail.com

Abstract

In the present work nonlocal problems with Bitsadze-Samarskii type conditions, with the first and the second kind integral conditions for mixed parabolic equation involving Riemann-Liouville fractional differential operator have been formulated and investigated. The uniqueness and the existence of the solution of the considered problems were proved. To do this, considered problems are equivalently reduced to the problems with nonlocal conditions with respect to the trace of the unknown function and its space-derivatives. Then using the representation of the solution of the second kind of Abel's integral equation, it was found integral representations of the solutions of these problems. Necessary conditions for the given functions were determined in order to provide a unique solvability of investigated problems.

Keywords: *mixed parabolic equation, nonlocal condition, fractional order equations, method of integral equations.*

Mathematics Subject Classification (2010): *35M10.*

Introduction

It is known that theory of boundary-value problems for fractional order partial differential equations is one of the intensively developing part of the general theory of partial differential equations.

In many real-life processes such as viscoelasticity [1], biosciences [2], diffusion processes [3] and dynamical processes in self-similar structures [4] and others, lead to differential equations of fractional order. For more details of fractional calculus, see, for example [5], [6], [7].

Omitting huge number of works, devoted to the studying problems for fractional order partial differential equations, we will mention publications that are very close to the present investigation. Precisely, in [8], using the method of Green-functions it has been obtained the general solution of the diffusion-wave equation and constructed solutions of the first, second and mixed boundary-value problems for diffusion-wave equation. In [9], using this method solutions of the main boundary value problems for time-fractional telegraph equation were obtained. The work [10] is devoted to study Gevrey problem for a loaded mixed parabolic equation involving Riemann-Liouville fractional derivative. For more information of problems for mixed parabolic equations, see, for example, [11], [12], [13] and references therein.

Studying nonlocal problems for fractional order equations is developing intensively. Precisely, in the works [14], [15],[16], [17] nonlocal problems were formulated and investigated for mixed type equations involving Riemann-Liouville and

Caputo fractional derivatives. And also the work [18] is devoted to well-posedness of boundary-value with nonlocal conditions for diffusion-wave equations.

We should note that the majority of the researches for mixed parabolic equations considered for equations with co-linear time directions, but equations with non-co-linear time directions studied a little.

In the present paper, parabolic equations with perpendicular time directions are considered and nonlocal problems are formulated and investigated for unique solvability.

1 Formulation of problems

In the domain $Q = Q_1 \cup Q_0 \cup Q_2$, we consider the following equation

$$0 = \begin{cases} L_1 u \equiv u_{xx}(x, t) - D_{0t}^\alpha u(x, t), & (x, t) \in Q_1, \\ L_2 u \equiv u_{tt}(x, t) + D_{x0}^\delta u(x, t), & (x, t) \in Q_2, \end{cases} \quad (1)$$

where $Q_1 = \{(x, t) : 0 < x < +\infty, 0 < t < T\}$, $Q_0 = \{(x, t) : x = 0, 0 < t < T\}$, $Q_2 = \{(x, t) : -h < x < 0, 0 < t < T\}$; α, δ, h, T are given positive real numbers, such that $0 < \alpha, \delta < 1$;

$$D_{mt}^\gamma [g(t)] = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_m^t |t-z|^{-\gamma} g(z) dz, \quad 0 < \gamma < 1,$$

is the Riemann-Liouville fractional derivative of order γ [6], $\Gamma(z)$ is Euler's gamma function [19].

$u(x, t)$ is called a regular solution of the equation (1), if it belongs to the class of functions $t^{1-\alpha} u(x, t) \in C(\overline{Q_1})$, $(-x)^{1-\delta} u(x, t) \in C(\overline{Q_2})$, $u_{xx}(x, t), D_{0t}^\alpha u(x, t) \in C(Q_1)$, $u_{tt}(x, t), D_{x0}^\delta u(x, t) \in C(Q_2)$ and satisfies equation (1) in $(x, t) \in Q_1 \cup Q_2$.

For the equation (1), we study the following problems in the domain Q :

Problem BS. (Problem with Bitsadze-Samarskii type conditions). Find a regular solution in the domain $Q_1 \cup Q_2$ of the equation (1), satisfying the following conditions

$$\lim_{t \rightarrow +0} D_{0t}^{\alpha-1} u(x, t) = \varphi_1(x), \quad 0 \leq x < +\infty; \quad (2)$$

$$u(x, 0) = \varphi_2(x), \quad -h < x < 0; \quad (3)$$

$$u(x, T) = a(x) u(x, \eta_0) + b(x), \quad -h < x < 0 \quad (4)$$

and gluing conditions

$$\lim_{x \rightarrow -0} (-x)^{1-\delta} u(x, t) = \lim_{x \rightarrow +0} u(x, t), \quad 0 \leq t \leq T; \quad (5)$$

$$\lim_{x \rightarrow -0} (-x)^{1-\delta} \frac{\partial}{\partial x} [(-x)^{1-\delta} u(x, t)] = \lim_{x \rightarrow +0} u_x(x, t), \quad 0 < t < T \quad (6)$$

$\varphi_1(x)$, $\varphi_2(x)$, $a(x)$ and $b(x)$ are given functions such that $a(x) \neq 0$, $x \in (-h, 0)$; $\eta_0 \in R$, $0 < \eta_0 < T$.

Problem I_1 (Problem with the first kind integral conditions). Find a regular solution in the domain $Q_1 \cup Q_2$ of the equation (1), satisfying conditions (2), (3) and

$$u(x, T) = a_1(x) \int_0^T u(x, t) dt + b_1(x), \quad -h < x < 0, \quad (7)$$

and the gluing conditions (5), (6), where $a_1(x)$, $b_1(x)$ are given functions, such that $a_1(x) \neq 0$, $x \in (-h; 0)$.

Problem I_2 (Problem with the second kind integral conditions). Find a regular solution in the domain $Q_1 \cup Q_2$ of the equation (1) satisfying conditions (2), (3) and

$$\int_0^T u(x, t) dt = \varphi_3(x), \quad -h < x < 0 \quad (8)$$

and the gluing conditions (5), (6), where $\varphi_3(x)$ is given function.

2 Investigation of the problem BS

The following theorem is valid.

Theorem 1. *Let the following conditions be fulfilled:*

$$\begin{aligned} \varphi_1(x) &= x^\varepsilon \tilde{\varphi}(x), \quad \varepsilon > (1 - 2\beta) / \beta, \quad \tilde{\varphi}(x) \in C[0, +\infty) \cap C^1(0, +\infty); \\ a(x) &\in C[-h; 0], \quad (-x)^{1-\delta} \varphi_2(x), \quad (-x)^{1-\delta} b(x) \in C[-h, 0], \quad \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_2(x) = 0; \end{aligned}$$

$$T \cdot E_{3-\beta, 2}(\lambda T^{3-\beta}) - a(0) \eta_0 E_{3-\beta, 2}(\lambda \eta_0^{3-\beta}) \neq 0.$$

$$E_{a,b}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(ak + b)}$$

is the Mittag-Leffler function of two parameters [5], $\beta = \alpha/2$.

Then there exists a unique solution of the problem BS.

Proof. Let $u(x, t)$ be a solution of the problem BS. Taking the conditions of the problem into account, we introduce the following notations and assumptions:

$$\lim_{x \rightarrow +0} u(x, t) = \lim_{x \rightarrow -0} (-x)^{1-\delta} u(x, t) = \tau(t), \quad \tau(t) \in C^1[0, T]; \quad (9)$$

$$\lim_{x \rightarrow +0} u_x(x, t) = \lim_{x \rightarrow -0} (-x)^{1-\delta} \frac{\partial}{\partial x} [(-x)^{1-\delta} u(x, t)] = \nu(t), \quad \nu(t) \in C[0, T]. \quad (10)$$

Then, the solution of the problem BS in the domain Q_1 can be represented in the form [10]

$$u(x, t) = - \int_0^t \nu(\eta) G(x, t; 0, \eta) d\eta + \int_0^{+\infty} \varphi_1(\xi) G(x, t; \xi, 0) d\xi, \quad (11)$$

where

$$G(x, t; \xi, \eta) = \frac{(t - \eta)^{\beta-1}}{2} \left[e_{1,\beta}^{1,\beta} \left(-\frac{|x - \xi|}{(t - \eta)^\beta} \right) + e_{1,\beta}^{1,\beta} \left(-\frac{|x + \xi|}{(t - \eta)^\beta} \right) \right], \beta = \alpha/2,$$

$$e_{1,\beta}^{1,\delta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n! \Gamma(\delta - \beta n)}$$

is Wright's function [5].

In formula (11) we pass to the limit as $x \rightarrow 0$ (as in the work [20]) and considering (10), we obtain the following functional relation between unknown functions $\tau(t)$ and $\nu(t)$:

$$\tau(t) = -\frac{1}{\Gamma(\beta)} \int_0^t \nu(\eta) (t - \eta)^{\beta-1} d\eta + \int_0^{+\infty} \varphi_1(\xi) G(0, t; \xi, 0) d\xi, \quad 0 < t < T. \quad (12)$$

Now, we consider problem BS in the domain Q_2 . Passing to the limit $x \rightarrow -0$ in the equation $L_2 u = 0$, we get

$$\tau''(t) + \Gamma(1 + \delta) \nu(t) = 0, \quad 0 < t < T. \quad (13)$$

Multiplying $(-x)^{1-\delta}$ to the both sides of the conditions (3) and (4), then passing to the limit at $x \rightarrow -0$ in taken equalities, considering (9), we obtain the following equalities

$$\tau(0) = \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_2(x), \quad \tau(T) = a(0) \tau(\eta_0) + \lim_{x \rightarrow -0} (-x)^{1-\delta} b(x). \quad (14)$$

Thus, the problem BS is equivalently reduced to find the solution satisfying conditions (14) of the system of equations $\{(12),(13)\}$ with respect to the unknown functions $\tau(t)$ and $\nu(t)$. So, we carry on our research to study the obtained problem.

If we temporarily assume that the function $\tau(t)$ is known in (12), then it will be Abel's integral equation with respect to unknown function $\nu(t)$ and it is easy to show that solvability condition of the equation is fulfilled. Then, using the formula of the solution of Abel's integral equation [21] from (12), we get

$$\nu(t) = -\frac{1}{\Gamma(1 - \beta)} \int_0^t \tau(\eta) (t - \eta)^{-\beta} d\eta + f_1(t), \quad (15)$$

where

$$f_1(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\eta)^{-\beta} \left\{ \int_0^{+\infty} \varphi_1(\xi) G(0, \eta; \xi, 0) d\xi \right\} d\eta.$$

Excluding function $\nu(t)$ from equalities (13) and (15), we obtain

$$\tau''(t) - \frac{\Gamma(1+\delta)}{\Gamma(1-\beta)} \int_0^t \tau(\eta) (t-\eta)^{-\beta} d\eta = -\Gamma(1+\delta) f_1(t), \quad t \in (0, T). \quad (16)$$

We replace t to z in (16), then integrate two times obtained equality with respect to z over $[0, t]$. As a result, introducing notation $\tau'(0) = C$ and taking into account equality $\tau(0) = 0$ (which follows from the conditions of the theorem 1), we get the following equation with respect to the unknown function $\tau(t)$

$$\tau(t) - \frac{\Gamma(1+\delta)}{\Gamma(3-\beta)} \int_0^t \tau(\eta) (t-\eta)^{2-\beta} d\eta = C \cdot t + \Gamma(1+\delta) \int_0^t f_1(\eta) (t-\eta) d\eta. \quad (17)$$

If we temporarily assume that C is known number in (17), then it will be the second kind Abel's integral equation with respect to $\tau(t)$ and its unique solution is defined in the form [6]

$$\begin{aligned} \tau(t) = & C \cdot \frac{d}{dt} \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \eta d\eta + \\ & + \lambda \frac{d}{dt} \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \left\{ \int_0^\eta f_1(z) (\eta-z) dz \right\} d\eta, \end{aligned} \quad (18)$$

where $\lambda = \Gamma(1+\delta)$.

First, differentiating the first summand of the equality (18), then using the rule of integration by parts, we get

$$\begin{aligned} \frac{d}{dt} \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \eta d\eta &= t + \int_0^t \frac{\partial}{\partial t} E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \cdot \eta d\eta = \\ &= t - \int_0^t \frac{\partial}{\partial \eta} E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \cdot \eta d\eta = \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] d\eta. \end{aligned}$$

Replacing the function $E_{3-\beta,1}(z)$ to its expand of power series as

$$E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] = \sum_{k=0}^{+\infty} \frac{[\lambda(t-\eta)^{3-\beta}]^k}{\Gamma[(3-\beta)k+1]} = \frac{1}{\Gamma(1)} + \frac{\lambda(t-\eta)^{3-\beta}}{\Gamma(4-\beta)} + \dots$$

then computing integral, we come

$$\frac{d}{dt} \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \eta d\eta = t \cdot E_{3-\beta,2} (\lambda t^{3-\beta}).$$

Considering $f_2(0) = 0$ and applying the rule of integration by parts to the second summand of (18), it is easy to show that

$$\frac{d}{dt} \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] \left[\int_0^\eta f_1(z) (\eta-z) dz \right] d\eta = \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] f'_1(\eta) d\eta.$$

Taking this into account, we rewrite the solution (18) in the form

$$\tau(t) = C \cdot t E_{3-\beta,2} (\lambda t^{3-\beta}) + \lambda \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] f'_1(\eta) d\eta. \quad (19)$$

For finding unknown number C , we substitute the solution (19) into the condition (14):

$$\begin{aligned} C \left[T \cdot E_{3-\beta,2} (\lambda T^{3-\beta}) - a(0) \eta_0 E_{3-\beta,2} (\lambda \eta_0^{3-\beta}) \right] &= \\ &= \lambda a(0) \int_0^{\eta_0} E_{3-\beta,1} [\lambda(\eta_0-\eta)^{3-\beta}] f'_1(\eta) d\eta - \\ &- \lambda \int_0^T E_{3-\beta,1} [\lambda(T-\eta)^{3-\beta}] f'_1(\eta) d\eta + \lim_{x \rightarrow -0} (-x)^{1-\delta} b(x). \end{aligned} \quad (20)$$

By virtue of the conditions of the theorem 1, we uniquely find unknown constant C from (20). Substituting taken value of C into (19), we define the solution $\tau(t)$ of the problem $\{(17), (14)\}$.

After finding $\tau(t)$ function, $\nu(t)$ function is defined by formula (15). After that the solution of the problem BS is represented by (11) in the domain Q_1 .

Let us introduce notation $u(x, \eta_0) = \varphi(x)$, $-h < x < 0$, then the solution of the problem BS can be written as a solution of the first boundary-value problem for the equation $L_2 u = 0$ in the domain Q_2 as follows [20]

$$\begin{aligned} u(x, t) &= \int_x^0 \varphi_2(\xi) \tilde{G}_\eta(x, t; \xi, 0) d\xi - \int_x^0 [a(\xi) \varphi(\xi) + b(\xi)] \tilde{G}_\eta(x, t; \xi, T) d\xi + \\ &+ \Gamma(\delta) \int_0^T \tau(\eta) \tilde{G}(x, t; 0, \eta) d\eta \end{aligned} \quad (21)$$

where

$$\begin{aligned} \tilde{G}(x, t; \xi, \eta) &= [(\xi - x)^{\theta-1}/2] \times \\ &\times \sum_{n=-\infty}^{+\infty} \left[e_{1,\theta}^{1,\theta} \left(-\frac{|t - \eta + 2nT|}{(\xi - x)^\theta} \right) - e_{1,\theta}^{1,\theta} \left(-\frac{|t + \eta + 2nT|}{(\xi - x)^\theta} \right) \right], \quad \theta = \delta/2. \end{aligned} \quad (22)$$

Substituting $t = \eta_0$ into (21), we get the following integral equation with respect to unknown function $\varphi(x)$:

$$\varphi(x) + \int_x^0 \varphi(\xi) K(x, \xi) d\xi = \Phi(x), \quad -h < x < 0, \quad (23)$$

where

$$\begin{aligned} K(x, \xi) &= a(\xi) \tilde{G}_\eta(x, \eta_0; \xi, T), \quad \Phi(x) = \int_x^0 \varphi_2(\xi) \tilde{G}_\eta(x, \eta_0; \xi, 0) d\xi - \\ &- \int_x^0 b(\xi) \tilde{G}_\eta(x, \eta_0; \xi, T) d\xi + \Gamma(\delta) \int_0^T \tau(\eta) \tilde{G}(x, \eta_0; 0, \eta) d\eta. \end{aligned}$$

We study the kernel of the equation (23). Firstly, we rewrite the function $\tilde{G}(x, t; \xi, \eta)$ as follows

$$\begin{aligned} \tilde{G}(x, t; \xi, \eta) &= \frac{(\xi - x)^{\theta-1}}{2} e_{1,\theta}^{1,\theta} \left(-\frac{|t - \eta|}{(\xi - x)^\theta} \right) - \frac{(\xi - x)^{\theta-1}}{2} e_{1,\theta}^{1,\theta} \left(-\frac{t + \eta}{(\xi - x)^\theta} \right) + \\ &+ \frac{(\xi - x)^{\theta-1}}{2} \sum_{n=1}^{+\infty} \left[e_{1,\theta}^{1,\theta} \left(-\frac{t - \eta + 2nT}{(\xi - x)^\theta} \right) - e_{1,\theta}^{1,\theta} \left(-\frac{t + \eta + 2nT}{(\xi - x)^\theta} \right) + \right. \\ &\left. + e_{1,\theta}^{1,\theta} \left(-\frac{2nT + \eta - t}{(\xi - x)^\theta} \right) - e_{1,\theta}^{1,\theta} \left(-\frac{2nT - \eta - t}{(\xi - x)^\theta} \right) \right]. \end{aligned}$$

Using the following

$$\frac{d}{dx} x^{\mu-1} e_{\alpha,\beta}^{\mu,\delta}(cx^\alpha) = x^{\mu-2} e_{\alpha,\beta}^{\mu-1,\delta}(cx^\alpha); \quad \frac{1}{z} e_{\alpha,\beta}^{-k,\delta}(z) = e_{\alpha,\beta}^{\alpha-k,\delta-\beta}(z), \quad k = 0, 1, 2, \dots$$

formulas [20], it is easy to verify that

$$\begin{aligned} \tilde{G}_\eta(x, \eta_0; \xi, T) &= \\ &= (\xi - x)^{-1} \sum_{n=0}^{+\infty} \left[e_{1,\theta}^{1,0} \left(-\frac{\eta_0 + (2n+1)T}{(\xi - x)^\theta} \right) - e_{1,\theta}^{1,0} \left(-\frac{-\eta_0 + (2n+1)T}{(\xi - x)^\theta} \right) \right]. \end{aligned}$$

Taking into account estimates for Wright function and $e_{\alpha,\beta}^{\mu,\delta}(-z)$ is a positive function [20], we get

$$|K(x, \xi)| \leq \left| a(\xi) \tilde{G}_\eta(x, \eta_0; \xi, T) \right| \leq C(\xi - x)^{\theta\rho-1}, \quad \rho \in (1; 2],$$

where C is a finite constant.

Similarly, taking into account properties of the given and $\tau(t)$ functions, it is easy to show that $\Phi(x) \in C[-h, 0]$. Thus, (23) is the second kind Volterra integral equation with weak singularity [21] with respect to the function $\varphi(x)$. By virtue of the properties of the kernel and the right-hand side of the equation, it follows that unique solution of the equation (23) exists and it belongs to the class of functions $C[-h, 0]$. Substituting finding function $\varphi(x)$ from (23) into (21), we find the solution of the problem BS in the domain Q_2 .

Theorem 1 has been proved. □

3 Investigation of the problem I_1

Theorem 2. *Let the following conditions be fulfilled:*

$$\begin{aligned} \varphi_1(x) &= x^\varepsilon \tilde{\varphi}(x), \quad \varepsilon > (1 - 2\beta) / \beta, \quad \tilde{\varphi}(x) \in C[0, +\infty) \cap C^1(0, +\infty); \\ a_1(x) &\in C[-h; 0], \quad (-x)^{1-\delta} \varphi_2(x), \quad (-x)^{1-\delta} b_1(x) \in C[-h; 0], \quad \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_2(x) = 0; \\ E_{3-\beta,1}(\lambda T^{3-\beta}) - a_1(0) T E_{3-\beta,3}(\lambda T^{3-\beta}) &\neq 0. \end{aligned}$$

Then there exists unique solution of the problem I_1 .

Proof. Let $u(x, t)$ be a solution of the problem I_1 . Based on the conditions of the problem I_1 , we introduce notations (9) and (10). Then, the solution of the problem I_1 can be written by formula (11) in the domain Q_1 . Passing to the limit as $x \rightarrow +0$ in (11), we find functional relation (12) between unknown functions $\tau(t)$ and $\nu(t)$. Now, we consider problem I_1 in the domain Q_2 , doing similar actions as the problem BS, we get equation (16) with respect to $\tau(t)$ function and the following conditions

$$\tau(0) = 0, \quad \tau(T) = a_1(0) \int_0^T \tau(t) dt + \lim_{x \rightarrow -0} (-x)^{1-\delta} b_1(x). \quad (24)$$

As it was shown at the problem BS, the solution of the equation (16), satisfying condition $\tau(0) = 0$, is defined by (19). For finding unknown number C in (19), substituting (19) to the second condition of (24), we obtain

$$C \cdot \left[T E_{3-\beta,2}(\lambda T^{3-\beta}) - a_1(0) \int_0^T t \cdot E_{3-\beta,2}(\lambda t^{3-\beta}) dt \right] = \lambda a_1(0) \times$$

$$\begin{aligned} & \times \int_0^T \left\{ \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] f'_1(\eta) d\eta \right\} dt - \lambda \int_0^T E_{3-\beta,1} [\lambda(T-\eta)^{3-\beta}] f'_1(\eta) d\eta + \\ & + \lim_{x \rightarrow -0} (-x)^{1-\delta} b_2(x). \end{aligned} \quad (25)$$

It is easy to show that, the following equalities are valid

$$\begin{aligned} & \int_0^T t \cdot E_{3-\beta,2} (\lambda t^{3-\beta}) dt = T^2 \cdot E_{3-\beta,3} (\lambda T^{3-\beta}); \\ & \int_0^T \left\{ \int_0^t E_{3-\beta,1} [\lambda(t-\eta)^{3-\beta}] f'_1(\eta) d\eta \right\} dt = \\ & = \int_0^T (T-\eta) \cdot E_{3-\beta,2} [\lambda(T-\eta)^{3-\beta}] f'_1(\eta) d\eta. \end{aligned}$$

Considering the last equalities, we rewrite (25) as follows

$$\begin{aligned} & C \cdot [TE_{3-\beta,1} (\lambda T^{3-\beta}) - a_1(0) T^2 E_{3-\beta,3} (\lambda T^{3-\beta})] = \\ & = \lambda a_1(0) \int_0^T f'_1(\eta) (T-\eta) E_{3-\beta,2} [\lambda(T-\eta)^{3-\beta}] d\eta - \\ & - \lambda \int_0^T E_{3-\beta,1} [\lambda(T-\eta)^{3-\beta}] f'_1(\eta) d\eta + \lim_{x \rightarrow -0} (-x)^{1-\delta} b_1(x). \end{aligned}$$

By virtue of the conditions of the theorem 2, from the last equality unknown number C is uniquely found. Substituting the obtained expression of the C into (19), we define $\tau(t)$ function for the problem I_1 .

Let us introduce notation $u(x, T) = \psi(x)$, $-h < x < 0$. Then, the solution of the problem I_1 is defined as a solution of the first boundary-value problem for the equation $L_2 u = 0$ in the domain Q_2 [20] in the following form

$$\begin{aligned} u(x, t) = & \int_x^0 \varphi_2(\xi) \tilde{G}_\eta(x, t; \xi, 0) d\xi - \int_x^0 \psi(\xi) \tilde{G}_\eta(x, t; \xi, T) d\xi + \\ & + \Gamma(\delta) \int_0^T \tau(\eta) \tilde{G}(x, t; 0, \eta) d\eta, \end{aligned} \quad (26)$$

where $\tilde{G}(x, t; \xi, \eta)$ is Green's function which is defined by (22).

Obeying the function (26) to the condition (7), changing order of integration in the obtained equality and taking into account notation $u(x, T) = \psi(x)$, we get the following equation

$$\psi(x) + \int_x^0 \psi(\xi) K_1(x, \xi) d\xi = \Phi_1(x), \quad -h < x < 0, \quad (27)$$

where

$$\begin{aligned} K_1(x, \xi) &= a_1(x) \int_0^T \tilde{G}_\eta(x, t; \xi, T) dt, \quad \Phi_1(x) = \\ &= a_1(x) \int_x^0 \varphi_2(\xi) d\xi \int_0^T \tilde{G}_\eta(x, t; 0, \eta) dt + \Gamma(\delta) \int_0^T \tau(\eta) d\eta \int_0^T \tilde{G}(x, t; 0, \eta) dt + b_1(x). \end{aligned}$$

We study properties of the kernel and the right-hand side of the equation (27). Form the view of the function $\tilde{G}(x, t; \xi, \eta)$, it follows that

$$\begin{aligned} \frac{\partial}{\partial \eta} \tilde{G}(x, t; \xi, \eta) &= -\frac{(\xi - x)^{\theta-1}}{2} \frac{\partial}{\partial t} \left\{ e_{1,\theta}^{1,\theta} \left(-\frac{|t - \eta|}{(\xi - x)^\theta} \right) + \right. \\ &+ \left. \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|t - \eta + 2nT|}{(\xi - x)^\theta} \right) + \sum_{n=-\infty}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|t + \eta + 2nT|}{(\xi - x)^\theta} \right) \right\}. \end{aligned}$$

Taking this into account, we have

$$\begin{aligned} \int_0^T \tilde{G}_\eta(x, t; \xi, T) dt &= -\frac{(\xi - x)^{\theta-1}}{2} \int_0^T \frac{\partial}{\partial t} e_{1,\theta}^{1,\theta} \left(-\frac{|t - \eta|}{(\xi - x)^\theta} \right) \Big|_{\eta=T} dt - \\ &- \frac{(\xi - x)^{\theta-1}}{2} \left\{ \int_0^T \frac{\partial}{\partial t} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|t + (2n - 1)T|}{(\xi - x)^\theta} \right) dt + \right. \\ &+ \left. \int_0^T \frac{\partial}{\partial t} \sum_{n=-\infty}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|t + (2n + 1)T|}{(\xi - x)^\theta} \right) dt \right\}. \end{aligned}$$

Using formulas [20]

$$\frac{1}{z} e_{\alpha,\beta}^{-k,\delta}(z) = e_{\alpha,\beta}^{\alpha-k,\delta-\beta}, \quad \lim_{z \rightarrow 0} \frac{1}{z} e_{\alpha,\beta}^{-k,\delta}(z) = \frac{1}{\Gamma(\alpha - k) \Gamma(\delta - \beta)}$$

it is easy to verify that the following

$$\int_0^T \tilde{G}_\eta(x, t; \xi, T) dt = -\frac{1}{\Gamma(\theta)} \frac{(\xi - x)^{\theta-1}}{2} + K_2(x, \xi), \quad (28)$$

equality is valid, where

$$\begin{aligned} K_2(x, \xi) = & \frac{(\xi - x)^{\theta-1}}{2} \left\{ e_{1,\theta}^{1,\theta} \left(-\frac{T}{(\xi - x)^\theta} \right) - \right. \\ & - \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|2nT|}{(\xi - x)^\theta} \right) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|(2n-1)T|}{(\xi - x)^\theta} \right) - \\ & \left. - \sum_{n=-\infty}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|(2n+2)T|}{(\xi - x)^\theta} \right) + \sum_{n=-\infty}^{+\infty} e_{1,\theta}^{1,\theta} \left(-\frac{|(2n+1)T|}{(\xi - x)^\theta} \right) \right\}. \end{aligned}$$

It is known that for the Wright's function the following inequality is true [20],

$$\left| x^{\mu-1} y^{\delta-1} e_{\alpha,\beta}^{\mu,\delta} \left(-\frac{x^\alpha}{y^\beta} \right) \right| \leq C \cdot x^{\mu-\alpha\rho-1} \cdot y^{\delta+\beta\rho-1}, \quad (29)$$

here $\theta \in (1, 2]$, if $\mu = 0$ or $\mu = 1$, C is a constant not depending on ρ .

Using this fact and the convergence of the generalized harmonic series, one can show that

$$\begin{aligned} |K_1(x, \xi)| &= |a_1(x)| \left| \int_0^T \tilde{G}_\eta(x, t; \xi, T) dt \right| = \\ &= |a_1(x)| \left| -\frac{1}{\Gamma(\theta)} \frac{(\xi - x)^{\theta-1}}{2} + K_2(x, \xi) \right| \leq M(\xi - x)^{\theta-1}, \end{aligned}$$

where M is positive finite real number.

Based on the last inequality, we conclude that $K_1(x, \xi) = (\xi - x)^{\theta-1} O(1)$. Similarly, it is easy to show that the right-hand side of the equation (27) belongs to the class $C[-h, 0]$.

Thus, (27) is the second kind Volterra integral equation with weak singularity. Then, according to the theory of Volterra integral equations [21], there exist solution of the equation (27) and this solution belongs to the class of functions $C[-h, 0]$. After finding $\psi(x)$ from (27), substituting it to (26), we define the solution of the problem I_1 in the domain Q_2 .

Theorem 2 has been proved. □

4 Investigation of the problem I_2

Theorem 3. *Let the following conditions be fulfilled:*

$$\begin{aligned} \varphi_1(x) &= x^\varepsilon \tilde{\varphi}(x), \varepsilon > (1 - 2\beta) / \beta, \tilde{\varphi}(x) \in C[0, +\infty) \cap C^1(0, +\infty); \\ (-x)^{1-\delta} \varphi_2(x) &\in C[-h, 0], \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_2(x) = 0, (-x)^{1-\delta} \varphi_3(x) \in C[-h; 0]. \end{aligned}$$

Then, there exists solution of the problem I_2 and it is unique.

Proof. Let $u(x, t)$ be a solution of the problem I_2 . Based on the conditions of the problem I_2 as in the problems above-studied, we introduce notations (9) and (10). Then, the solution of the problem 3 can be written by formula (11) in the domain Q_1 . Doing similar actions in the previous problems, passing to the limit as $x \rightarrow +0$ in (11), we get functional relation (12) between unknown functions $\tau(t)$ and $\nu(t)$. Now, considering the problem I_2 in the domain Q_2 , similarly as in problem BS, we obtain equation (16) with respect to $\tau(t)$ and the following

$$\tau(0) = 0, \int_0^T \tau(t) dt = \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_3(x) \quad (30)$$

conditions.

As it was shown that the solution of the equation (16) satisfying condition $\tau(0) = 0$, is written by (19). Obeying solution (19) to the second condition of (30), we find unknown C as follows:

$$C = \left\{ \lim_{x \rightarrow -0} (-x)^{1-\delta} \varphi_3(x) - \lambda \int_0^T f'_2(\eta) (T - \eta) E_{\alpha,2}[\lambda(T - \eta)^\alpha] d\eta \right\} [T^2 E_{\alpha,3}(\lambda T^\alpha)]^{-1}.$$

Now, we introduce notation $u(x, T) = \psi(x)$, $x \in (-h; 0)$. Then, the solution of the problem I_2 can be written in the form (26) in the domain Q_2 [4].

Substituting (26) into the condition (8), we have

$$\begin{aligned} &\int_0^T \int_x^0 \psi(\xi) \tilde{G}_\eta(x, t; \xi, T) d\xi dt = \\ &= \int_0^T \int_x^0 \varphi_2(\xi) \tilde{G}_\eta(x, t; \xi, 0) d\xi dt + \Gamma(\delta) \int_0^T \int_0^T \tau(\eta) \tilde{G}(x, t; 0, \eta) d\eta dt + \varphi_3(x). \end{aligned}$$

After some evaluations and taking (28) into account, we rewrite the last equality in the form

$$\frac{1}{\Gamma(\theta)} \int_x^0 \psi(\xi) (\xi - x)^{\theta-1} d\xi = 2 \int_x^0 \psi(\xi) K_2(x, \xi) d\xi + \Phi_2(x), \quad (31)$$

where

$$\Phi_2(x) = -2 \int_x^0 \varphi_2(\xi) d\xi \int_0^T \tilde{G}_\eta(x, t; \xi, 0) dt - 2\Gamma(\delta) \int_0^T \tau(\eta) d\eta \int_0^T \tilde{G}(x, t; 0, \eta) dt - 2\varphi_3(x).$$

If we temporarily assume that the right-hand side of (31) is a known function, then it will be Abel's integral equation with respect to unknown function $\psi(x)$ and its solution is written as follows [21]:

$$\begin{aligned} \psi(x) = & -\frac{2}{\Gamma(1-\theta)} \frac{d}{dx} \int_x^0 (t-x)^{-\theta} \left\{ \int_t^0 \psi(\xi) K_2(t, \xi) d\xi \right\} dt - \\ & -\frac{1}{\Gamma(1-\theta)} \frac{d}{dx} \int_x^0 (t-x)^{-\theta} \Phi_2(t) dt. \end{aligned} \quad (32)$$

We denote the first summand of the right-hand side of (32) by J and simplify it. For this aim we rewrite J as follows

$$J = -\frac{1}{\Gamma(1-\theta)} \frac{d}{dx} \int_x^0 \psi(\xi) d\xi \int_x^\xi (t-x)^{-\theta} K_2(t, \xi) dt.$$

Firstly, we apply the rule of integration by parts to the inner integral, after that differentiate with respect to x

$$J = -\frac{1}{\Gamma(1-\theta)} \int_x^0 \psi(\xi) d\xi \int_x^\xi (t-x)^{-\theta} \frac{\partial}{\partial t} K_2(t, \xi) dt.$$

Substituting obtained expression of J into (32), we get the following equation for the function $\psi(x)$

$$\psi(x) + \int_x^0 \psi(\xi) K_3(x, \xi) d\xi = \Phi_3(x), \quad -h < x < 0, \quad (33)$$

where

$$\begin{aligned} K_3(x, \xi) = & \frac{2}{\Gamma(1-\theta)} \int_x^\xi (t-x)^{-\theta} (\partial/\partial t) K_2(t, \xi) d\xi, \\ \Phi_3(x) = & -\frac{1}{\Gamma(1-\theta)} \frac{d}{dx} \int_x^0 (t-x)^{-\theta} \Phi_2(t) dt. \end{aligned}$$

Let us consider the kernel $K_3(x, \xi)$ of the equation (33). Using representation of the function $K_2(x, \xi)$ and differentiating formulas for Wright's function [20], it is easy to show that

$$\begin{aligned} \frac{\partial}{\partial t} K_2(t, \xi) &= (\xi - t)^{\theta-2} \left\{ e_{1,\theta}^{1,\theta-1} \left(-\frac{T}{(\xi - t)^\theta} \right) - \right. \\ &- \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \left[e_{1,\theta}^{1,\theta-1} \left(-\frac{|2nT|}{(\xi - t)^\theta} \right) - e_{1,\theta}^{1,\theta-1} \left(-\frac{|(2n-1)T|}{(\xi - t)^\theta} \right) \right] - \\ &\left. - \sum_{n=-\infty}^{+\infty} \left[e_{1,\theta}^{1,\theta-1} \left(-\frac{|(2n+2)T|}{(\xi - t)^\theta} \right) - e_{1,\theta}^{1,\theta-1} \left(-\frac{|(2n+1)T|}{(\xi - t)^\theta} \right) \right] \right\}. \end{aligned}$$

This function has an arbitrary order derivatives for $t \neq \xi$ and this function and its derivatives tend zero at $t \rightarrow \xi$, i.e. they are bounded.

On the basis of the conditions of the theorem 3, one can show that $\Phi_3(x) \in C[-h, 0]$. It follows that, (33) is a second kind Volterra integral equation with respect to unknown function $\psi(x)$ and according to the theory of integral equations there exists unique solution of this equation belonging to the class $C[-h, 0]$.

After finding unknown function $\psi(x)$ from (34), the solution of the problem in the domain Q_2 is defined by (26) and in the domain Q_1 by formula (11).

Theorem 3 has been proved. □

References

- [1] Bagley R.L. and Torvik P.J. A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology*, Vol. 27, 201–210 (1983).
- [2] Magin R. Fractional calculus in bioengineering. *Crit. Rev. Biom. Eng.*, Vol. 32, Issue 1, 1–104 (2004).
- [3] Metzler R., Klafter J. Boundary value problems for fractional diffusion equations. *Physica A* 278, 107–125 (2000).
- [4] Mainardi F. *Fractals and Fractional Calculus in Continuum Mechanics*. Springer, New York (1997).
- [5] Podlubny I. *Fractional Differential Equations*. Academic Press, New York (1999).
- [6] Samko S.G., Kilbas A.A., Marichev O.I. *Fractional Integrals and Derivatives*. Gordon and Breach Science Publishers, London (1993).
- [7] Kilbas A.A., Srivastava H.M., Trujillo J.J. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Vol. 204, Elsevier Science B. V., Amsterdam (2006).

- [8] Pskhu A.V. Solution of bounadry value problems for the fractional diffusion equation by the Green function method. *Differential equations*, Vol. 39, 1509–1513 (2003).
- [9] Mamchuev M.O. Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method. *Fractional Calculus and Applied Analysis*, Vol. 20, Issue 1, 190–211 (2017).
- [10] Gekkieva S.Kh. Gevrey problem for a loaded mixed parabolic equation with a fractional derivative. *Journal of Mathematical Sciences*, Vol. 250, No. 5, 746–752 (2020).
- [11] Popov S.V. The Gevrey boundary value problem for a third order equation. *Mat. Zam. Sev.-Zap. Fed. Univ.*, Vol. 24, Issue 1, 43–56 (2017).
- [12] Tersenov S.A. *Parabolic Equations with Changing Direction of Time*. Nauka, Novosibirsk, (1985). [in Russian].
- [13] Mamanazarov A.O. Unique Solvability of Problems for a Mixed Parabolic Equation in Unbounded Domain. *Lobachevskii Journal of Mathematics*, Vol. 41, No. 9, 1830–1838 (2020).
- [14] Salakhitdinov M.S., Karimov E.T. Uniqueness of an inverse source non-local problem for fractional order mixed type equations. *Eurasian Mathematical Journal*, Vol. 7, Issue 1, 74–83 (2016).
- [15] Kerbal S., Kadirkulov B.J., Turmetov B. Solvability of a nonlocal boundary value problem involving fractional derivative operators. *Math. Model. Nat. Phenom.*, Vol. 12, Issue 3, 72–81 (2017).
- [16] Kerbal S., Karimov E., Rakhmatullaeva N. Non-local Boundary Problem with Integral Form Transmitting Condition for Fractional Mixed Type Equation in a Composite Domain, *Math. Model. Nat. Phenom.* Vol. 12, Issue 3, 95–104 (2017).
- [17] Agarwal P., Berdyshev A., Karimov E. Solvability of a Non-local Problem with Integral Transmitting Condition for Mixed Type Equation with Caputo Fractional Derivative. *Results in Mathematics*, Vol. 71, 1235–1257 (2017).
- [18] Mamchuev M.O. On the well-posedness of boundary value problems for a fractional-wave equation and one approach to solving them. *Differential equations*, Vol. 56, Issue 6, 756–760 (2020).
- [19] Bateman G. and Erdelyi A. *Higher Transcendental Functions. Hypergeometric Functions. Legendre Functions*. McGraw-Hill, New York (1953).
- [20] Pskhu A.V. *Fractional Partial Differential Equations*. Nauka, Moscow (2005). [in Russian].

- [21] Mikhlin S.G. Linear Integral Equations Fizmatgiz. Moscow (1959); Hindustan (1960); Dover, New York (2020).