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ON TIME-OPTIMAL CONTROL PROBLEM ASSOCIATED WITH PARABOLIC EQUATION

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Abstract

The boundary control problem for heat equation in a right rectangle domain is considered. The control parameter is equal to the temperature on some part of the border of the considered domain The estimate of a minimal time for achieving the given average temperature over some subdomain is found.

Keywords: *Minimal time, integral equation, boundary control, initial-boundary problem, admissible control.*

Mathematics Subject Classification (2010): *35K05, 35K15.*

Introduction

Consider the following mathematical model of the heat conduction process along the domain $\Omega = \{(x,y) \in R^2 : 0 < x < a,\, 0 < y < b\}$:

$$u_t = u_{xx} + u_{yy}, \quad (x,y) \in \Omega \quad t > 0, \tag{1}$$

with boundary conditions

$$u|_{x=0} \; = \; \varphi(y)\mu(t), \quad u|_{x=a} \; = \; 0, \quad 0 < x < a, \tag{2}$$

$$u|_{y=0} \; = \; 0, \quad u|_{y=b} \; = \; 0, \quad 0 < y < b, \quad t > 0, \tag{3}$$

and initial condition

$$u(x,y,0) \; = \; 0, \quad 0 \le x \le a, \quad 0 \le y \le b. \tag{4}$$

Let $M > 0$ be some given constant. We say that the function $\mu(t)$ is an *admissible control* if this function is differentiable on the half-line $t \ge 0$ and satisfies the following constraints

$$\mu(0) = 0, \quad |\mu(t)| \; \le \; M, \quad t > 0. \tag{5}$$

Assume that the function $\varphi(y) \in W_2^2[0,b]$ is smooth and satisfies conditions

$$\varphi(0) \; = \; \varphi(b) \; = \; 0, \quad \varphi_m \ge 0, \quad 0 \le y \le b, \tag{6}$$

where

$$\varphi_m = \frac{2}{b} \int\limits_0^b \varphi(y) \sin \frac{m\pi y}{b} \, dy. \tag{7}$$

Set of functions which satisfies conditions (6) is not empty. For example:

$$\varphi(y) \;=\; y\,(b-y) \;=\; \sum_{m=1}^{\infty} \varphi_m \sin\frac{m\pi y}{b},\quad 0 \le y \le b.$$

In the present work we consider the following problem.

Problem A. *Given constants α, $\beta \ge 1$ and $\theta > 0$ Problem A consists in looking for the minimal value of $T > 0$ so that for $t > 0$ the solution $u(x,y,t)$ of the initial-boundary value problem (1)-(4) with some admissible control $\mu(t)$ exists and for all $t \ge T$ satisfies the equation*

$$\int_0^{b/\alpha}\int_0^{a/\beta} u(x,y,t)\,dx\,dy \;=\; \theta,\quad t \ge T. \tag{8}$$

We recall that the time-optimal control problem for partial differential equations of parabolic type was first concerned in [6] and [9]. More recent results concerned with this problem were established in [1]-[5], [7], [8], [15], [16]. Detailed information on the problems of optimal control for distributed parameter systems is given in the monographs [10] and [14].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [13].

To formulate the main result we use some data related to the conditions of the boundary-initial problem (1)-(4).

Set

$$\rho_{nm} \;=\; \frac{8\,b\,\varphi_m}{a\,m\,\pi}\sin^2\frac{n\pi}{2\beta}\sin^2\frac{m\pi}{2\alpha},\quad n,\,m = 1,2,3,... \tag{9}$$

Theorem 0.1. *Let*

$$0 < \theta < \frac{\rho_{11}\,M\,a^2\,b^2}{\pi^2\,(a^2+b^2)}.$$

Set

$$T_0 = -\frac{a^2\,b^2}{\pi^2\,(a^2+b^2)}\ln\left(1 - \frac{\theta\,\pi^2\,(a^2+b^2)}{\rho_{11}\,M\,a^2\,b^2}\right).$$

Then a solution T_{min} of the Problem A exists and the estimate $T_{min} \le T_0$ is valid.

1 The main integral equation

Let B be the Banach space and $T > 0$. Denote by $C([0,T] \to B)$ the Banach space of all continuous maps $u : [0,T] \to B$ with the norm

$$\|u\| = \max_{0\le t\le T}\|u(t)\|.$$

By symbol $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions whose trace is equal to $\partial\Omega$ zero. Note that since $\widetilde{W}_2^1(\Omega)$ is closed, the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in metric $W_2^1(\Omega)$ also belongs to $\widetilde{W}_2^1(\Omega)$.

Definition 1.1. *By the solution of problem* (1) - (4) *we mean function* $u(x,y,t)$, *represented in the form*

$$u(x,y,t) = \frac{a-x}{a}\,\varphi(y)\,\mu(t) - v(x,y,t),$$

where the function $v(x,y,t)$ *is a generalized solution from* $C([0,T] \to \widetilde{W}_2^1(\Omega))$ *of the problem*

$$v_t(x,y,t) - \Delta v(x,y,t) \;=\; -\frac{a-x}{a}\,\varphi''(y)\,\mu(t) + \frac{a-x}{a}\,\varphi(y)\,\mu'(t),$$

with boundary conditions

$$v(x,y,t)\mid_{\partial\Omega} \;=\; 0,$$

and initial condition

$$v(x,y,0) = 0, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

Consequently,

$$v(x,y,t) \;=\; \int\limits_0^t \sum_{n,m=1}^{\infty} e^{-\lambda_{nm}(t-s)}\,\frac{\varphi_m}{n}\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}\left(\frac{2\pi m^2}{b^2}\mu(s) + \frac{2}{\pi}\mu'(s)\right)ds,$$

where $\lambda_{nm} = (n\pi/a)^2 + (m\pi/b)^2$, $n,m = 1,2,...$ (see, e.g. [17], [18]).

Note that the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^1(\Omega)$ considered in the monograph [11] in order to define a problem with homogeneous boundary conditions. Thus, the generalized solution introduced above is also a generalized solution in the sense of monograph [11]. However, unlike a solution from the class $W_2^1(\Omega)$, which is guaranteed to have a trace of almost all $t \in [0,T]$, a solution from the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ continuously depends of $t \in [0,T]$ in the metric $L_2(\Omega)$.

Proposition 1.1. *Let* $\mu(t)$ *be a smooth function on the half-line* $t \ge 0$ *and* $\varphi \in W_2^2[0,b]$. *Then the function*

$$u(x,y,t) \;=\; \int\limits_0^t \mu(s) \sum_{n,m=1}^{\infty} \frac{2\pi\, n\, \varphi_m}{a^2} e^{-\lambda_{nm}(t-s)}\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}\,ds, \tag{10}$$

is the solution of the initial-boundary value problem (1)-(4).

Proof. A similar proof is given in the case [12]. We rewrite the solution to the problem in the form

$$u(x,y,t) = \frac{a-x}{a}\,\varphi(y)\,\mu(t) -$$

$$-\int\limits_0^t \sum_{n,m=1}^{\infty} e^{-\lambda_{nm}(t-s)}\,\frac{\varphi_m}{n}\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}\left(\frac{2\pi m^2}{b^2}\mu(s) + \frac{2}{\pi}\mu'(s)\right)ds.$$

We show that function $v(x,y,t)$ belongs to class $C([0,T] \to \widetilde{W}_2^1(\Omega))$. For this, it is enough to prove that the gradient of this function, taken in $(x,y) \in \Omega$, continuously depends on $t \in [0,T]$ in the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is

$$\|\nabla v\|^2_{L_2(\Omega)} = \sum_{n,m=1}^{\infty} \frac{|\varphi_m|^2}{n^2} \lambda_{nm}\, b^2_{nm}(t)$$

where

$$b_{nm}(t) = \int_0^t e^{-\lambda_{nm}(t-s)} \left(\frac{2\pi m^2}{b^2} \mu(s) + \frac{2}{\pi} \mu'(s) \right) ds.$$

From the Cauchy-Bunyakovsky inequality, we obtain the following estimate

$$b_{nm}(t) \le \frac{C_1}{\sqrt{\lambda_{nm}}} + C_2 \frac{m^2}{\lambda_{nm}} \le C_3 \frac{m}{\sqrt{\lambda_{nm}}}, \quad t \ge 0.$$

From (6) and (7), we write

$$\varphi_m \;=\; \frac{2}{b} \int_0^b \varphi(y)\, \sin \frac{m\pi y}{b}\, dy \;=\; -\,\frac{2}{b}\, \varphi(y) \frac{b}{m\pi} \cos \frac{m\pi y}{b} \bigg|_{y=0}^{y=b} \;+$$

$$+\; \frac{2}{m\pi} \int_0^b \varphi'(y) \cos \frac{m\pi y}{b}\, dy \;=\; \frac{2}{m\pi} \int_0^b \varphi'(y) \cos \frac{m\pi y}{b}\, dy \;=\; \frac{b}{m\,\pi} \varphi'_m. \tag{11}$$

Consequently,

$$\|\nabla v\|^2_{L_2(\Omega)} \le C_3 \sum_{n,m=1}^{\infty} \frac{m^2 |\varphi_m|^2}{n^2} \le C_3 \frac{\pi^2}{6} \frac{b^2}{\pi^2} \sum_{m=1}^{\infty} m^2 \frac{|\varphi'_m|^2}{m^2} = C \|\varphi'\|^2_{L_2[0,b]}.$$

□

It is easy to see that with the equality (10) we can write the condition (8) of problem as the following

$$\theta \;=\; \int_0^{b/\alpha} \int_0^{a/\beta} u(x,y,t)\, dx\, dy \;=$$

$$= \int_0^t \mu(s) ds \int_0^{b/\alpha} \int_0^{a/\beta} \sum_{n,m=1}^{\infty} \frac{2\,\pi\, n}{a^2}\, \varphi_m\, e^{-\lambda_{nm}(t-s)}\, \sin \frac{n\pi x}{a}\, \sin \frac{m\pi y}{b}\, dx\, dy =$$

$$= \int_0^t \mu(s) \sum_{n,m=1}^{\infty} \frac{8b\, \varphi_m}{a\, m\, \pi}\, e^{-\lambda_{nm}(t-s)}\, \sin^2 \frac{n\pi}{2\beta} \sin^2 \frac{m\pi}{2\alpha}\, ds. \tag{12}$$

Set

$$B(t) = \sum_{n,m=1}^{\infty} \rho_{nm}\, e^{-\lambda_{nm} t}, \tag{13}$$

where ρ_{nm} defined by (9).

Then we get main integral equation

$$\int_0^t B(t-s)\mu(s)ds = \theta, \quad t > 0. \tag{14}$$

Proposition 1.2. *For $B(t)$ defined by (13) the following estimate*

$$0 < B(t) \leq \frac{C}{\sqrt{t}}, \quad 0 < t \leq 1, \tag{15}$$

is valid.

Proof. From (9) and (11), we may write

$$0 < \rho_{nm} \leq \frac{C_0\, \varphi'_m}{m^2},$$

Hence, using the definition (13) we get

$$0 < B(t) \leq C_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\varphi'_m}{m^2}\, e^{-\lambda_{nm} t} = C_0 \sum_{m=1}^{\infty} \frac{\varphi'_m}{m^2}\, e^{-\frac{\pi^2}{b^2} m^2 t} \sum_{n=1}^{\infty} e^{-\frac{\pi^2}{a^2} n^2 t}. \tag{16}$$

Set

$$K(t) = \sum_{m=1}^{\infty} \frac{\varphi'_m}{m^2}\, e^{-\frac{\pi^2}{b^2} m^2 t}, \quad t > 0.$$

This function for any $T > 0$ and $0 \leq t \leq T$ satisfies inequalities

$$0 < K(T) \leq K(t) \leq K(0). \tag{17}$$

For any $p > 0$ consider the following relations:

$$\sum_{n=1}^{\infty} e^{-pn^2} = \sum_{n=1}^{\infty} \int_n^{n+1} e^{-p[s]^2} ds = \int_1^{\infty} e^{-p[s]^2} ds = \int_1^{\infty} e^{-ps^2}\, e^{p(s^2-[s]^2)} ds,$$

where $[s]$ is integer part of s.

Note that $e^{p(s^2-[s]^2)} = e^{p(s-[s])(s+[s])} \leq e^{2ps}$. Then we obtain

$$\int_1^{\infty} e^{-ps^2}\, e^{p(s^2-[s]^2)} ds \leq \int_1^{\infty} e^{-ps^2+2ps} ds = e^p \int_1^{\infty} e^{-p\,(s-1)^2}\, ds.$$

Hence, for $0 < p \leq \text{const}$ we get

$$\sum_{n=1}^{\infty} e^{-pn^2} \leq \int_1^{\infty} e^{-ps^2}\, e^{p(s^2-[s]^2)}\, ds \leq e^p \int_0^{\infty} e^{-ps^2} ds \leq \frac{C}{\sqrt{p}}. \tag{18}$$

Put $p = \dfrac{\pi^2 t}{a^2}$. Then required estimate (15) follows from (16)-(18).

□

2 Estimate of Minimal Time

We consider the following integral equation

$$\int_0^t B(t-s)\,\mu(s)\,ds \;=\; \theta, \quad t \geq T, \tag{19}$$

where

$$B(t) = \sum_{n,m=1}^{\infty} \rho_{nm}\, e^{-\lambda_{nm} t}. \tag{20}$$

Proposition 2.1. *For the function defined by equality* (20) *the following estimate*

$$B(t) \geq \rho_{11}\, e^{-[(\pi/a)^2 + (\pi/b)^2]t}, \tag{21}$$

is valid.

Proof. The proof comes from functional series defined by (20) is non-negative. □

We introduce a specific heating as

$$Q(t) = \int_0^t B(t-s)ds = \int_0^t B(s)ds = \sum_{n,m=1}^{\infty} \frac{\rho_{nm}}{\lambda_{nm}} \left(1 - e^{-\lambda_{nm} t}\right). \tag{22}$$

The physical meaning of this function is evident: $Q(t)$ equals the average temperature of Ω in case where the heater is acting unit load (see, e.g. [1], [2]).

It is clear that $Q(0) = 0$ and $Q'(t) = B(t) \geq 0$.

Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^{\infty} B(s)ds. \tag{23}$$

Obviously, the average temperature of Ω in the case where the heater is acting with unit load cannot exceed Q^*.

Proposition 2.2. *Let*

$$0 < \theta < MQ^*. \tag{24}$$

Then there exist $T > 0$ and a real-valued measurable function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality

$$\int_0^T B(T-s)\mu(s)ds = \theta \tag{25}$$

is valid.

Proof. This follows from the properties of the function Q. Indeed, if we set $\mu(t) = M$ then

$$\int_0^t B(t-s)\mu(s)ds = M\int_0^t B(t-s)ds = MQ(t),$$

and because of (25) there exists $T > 0$ so that $MQ(T) = \theta$.

□

Remark 2.1. *It is clear that the value T, which was found in Proposition 2.2, gives a solution to the problem. Namely, T is the root of the equation*

$$Q(T) = \frac{\theta}{M}. \tag{26}$$

However, the main idea of the present work is to establish an acceptable estimate for the value of the minimal time T (see, e.g. [3]).

Proposition 2.3. *Let*

$$0 < \theta < \frac{\rho_{11}\, M\, a^2\, b^2}{\pi^2(a^2+b^2)}. \tag{27}$$

Then there exists $T > 0$ so that

$$T < -\frac{a^2\, b^2}{\pi^2\,(a^2+b^2)}\ln\left(1 - \frac{\theta\,\pi^2\,(a^2+b^2)}{\rho_{11}\, M\, a^2\, b^2}\right) \tag{28}$$

and the equality (26) *is fulfilled.*

Proof. For obtaining the required estimate we use Proposition 2.1. We may write

$$Q(t) = \int_0^t B(s)ds \geq \rho_{11}\int_0^t e^{-[(\pi/a)^2+(\pi/b)^2]s}ds =$$

$$= \frac{\rho_{11}\, a^2\, b^2}{\pi^2\,(a^2+b^2)}\left(1 - e^{-[(\pi/a)^2+(\pi/b)^2]t}\right). \tag{29}$$

Consider the following equation for the defining of T_0:

$$\frac{\rho_{11}\, a^2\, b^2}{\pi^2\,(a^2+b^2)}\left(1 - e^{-[(\pi/a)^2+(\pi/b)^2]T_0}\right) = \frac{\theta}{M}. \tag{30}$$

Then

$$T_0 = -\frac{a^2\, b^2}{\pi^2\,(a^2+b^2)}\ln\left(1 - \frac{\theta\,\pi^2\,(a^2+b^2)}{\rho_{11}\, M\, a^2\, b^2}\right).$$

In accordance with (29) and (30) we may write

$$0 < \frac{\theta}{M} \leq Q(T_0).$$

Then obviously there exists T, $0 < T < T_0$, which is a solution to the equation (26). □

Proposition 2.4. *Let $T > 0$ satisfies the equality (26) and condition (27).*

Then there exist $T_1 > T$ and a measurable real-valued function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality

$$\int_0^{b/\alpha}\int_0^{a/\beta} u(x,y,t)\,dx\,dy = \theta, \quad T \leq t \leq T_1,$$

is valid.

Proof. According to the following

$$\int_0^t B(t-s)\mu(s)ds = \theta, \tag{31}$$

it is enough to prove that there exists solution of the equation

$$\int_0^t B(t-s)\mu(s)ds = f(t), \quad 0 \leq t \leq T_1, \tag{32}$$

where

$$f(t) = \begin{cases} MQ(t), & \text{if } 0 \leq t \leq T, \\ \theta, & \text{if } T < t \leq T_1. \end{cases} \tag{33}$$

The solution (33) is piecewise smooth and, according to equality (26), is continuous.

Set

$$\mu(t) = \begin{cases} M, & \text{if } 0 \leq t \leq T, \\ \mu_1(t), & \text{if } T < t \leq T_1, \end{cases} \tag{34}$$

where $\mu_1(t)$ is the solution of the following integral equation

$$\int_0^T B(t-s)Mds + \int_T^t B(t-s)\mu_1(s)ds = \theta, \quad T \leq t \leq T_1. \tag{35}$$

After differentiating this equation we get

$$B(0)\mu_1(t) + \int_T^t B'(t-s)\mu_1(s)ds = M[B(t-T) - B(t)]. \tag{36}$$

According to (20) $B(0)$ positive and $B(t)$ function is convergence function on given interval. Hence, equation (36) has a unique solution $\mu_1(t)$ for all $t \geq T$, which is continuous function on the half line $t \geq T$. Besides,

$$\mu_1(T) = M\left(1 - \frac{B(T)}{B(0)}\right) < M,$$

and there exists $T_1 > T$ so that

$$|\mu_1(t)| \le M, \quad T \le t \le T_1.$$

It is clear that this function is the unique solution of the equation (35). Hence, the function (34) is piecewise continuous and satisfies equation (32). Consequently, this function $\mu(t)$, which has a jump at the point $t = T$, is the required solution.

$\square$

The proof of Theorem 0.1 follows now easily from Proposition 2.3 and Proposition 2.4.

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