On time-optimal control problem associated with parabolic equation

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Abstract

The boundary control problem for heat equation in a right rectangle domain is considered. The control parameter is equal to the temperature on some part of the border of the considered domain. The estimate of a minimal time for achieving the given average temperature over some subdomain is found.

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Introduction

Consider the following mathematical model of the heat conduction process along the domain \( \Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\} \):

\[
    u_t = u_{xx} + u_{yy}, \quad (x, y) \in \Omega, \quad t > 0,
\]

with boundary conditions

\[
    u\mid_{x=0} = \varphi(y)\mu(t), \quad u\mid_{x=a} = 0, \quad 0 < x < a, \quad t > 0,
\]

\[
    u\mid_{y=0} = 0, \quad u\mid_{y=b} = 0, \quad 0 < y < b, \quad t > 0,
\]

and initial condition

\[
    u(x, y, 0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.
\]

Let \( M > 0 \) be some given constant. We say that the function \( \mu(t) \) is an admissible control if this function is differentiable on the half-line \( t \geq 0 \) and satisfies the following constraints

\[
    \mu(0) = 0, \quad |\mu(t)| \leq M, \quad t > 0.
\]

Assume that the function \( \varphi(y) \in W^2_2[0, b] \) is smooth and satisfies conditions

\[
    \varphi(0) = \varphi(b) = 0, \quad \varphi_m \geq 0, \quad 0 \leq y \leq b,
\]

where

\[
    \varphi_m = \frac{2}{b} \int_{0}^{b} \varphi(y) \sin \frac{m\pi y}{b} \, dy.
\]
Set of functions which satisfies conditions (6) is not empty. For example:

\[ \varphi(y) = y(b-y) = \sum_{m=1}^{\infty} \varphi_m \sin \frac{m\pi y}{b}, \quad 0 \leq y \leq b. \]

In the present work we consider the following problem.

**Problem A.** Given constants \( \alpha, \beta \geq 1 \) and \( \theta > 0 \) Problem A consists in looking for the minimal value of \( T > 0 \) so that for \( t > 0 \) the solution \( u(x, y, t) \) of the initial-boundary value problem (1)-(4) with some admissible control \( \mu(t) \) exists and for all \( t \geq T \) satisfies the equation

\[ \int_{0}^{b/a} \int_{0}^{a/\beta} u(x, y, t) \, dx \, dy = \theta, \quad t \geq T. \]  

(8)

We recall that the time-optimal control problem for partial differential equations of parabolic type was first concerned in [6] and [9]. More recent results concerned with this problem were established in [1]-[5], [7], [8], [15], [16]. Detailed information on the problems of optimal control for distributed parameter systems is given in the monographs [10] and [14].

General numerical optimization and optimal boundary control have been studied in a great number of publications such as [13].

To formulate the main result we use some data related to the conditions of the boundary-initial problem (1)-(4).

Set

\[ \rho_{nm} = \frac{8b}{a \pi} \varphi_m \sin^2 \frac{n\pi}{2} \sin^2 \frac{m\pi}{2\alpha}, \quad n, m = 1, 2, 3, ... \]  

(9)

**Theorem 0.1.** Let

\[ 0 < \theta < \rho_{11} M a^2 b^2 / \pi^2 (a^2 + b^2). \]

Set

\[ T_0 = -\frac{a^2 b^2}{\pi^2 (a^2 + b^2)} \ln \left( 1 - \frac{\theta^2 (a^2 + b^2)}{\rho_{11} M a^2 b^2} \right). \]

Then a solution \( T_{\text{min}} \) of the Problem A exists and the estimate \( T_{\text{min}} \leq T_0 \) is valid.

1 The main integral equation

Let \( B \) be the Banach space and \( T > 0 \). Denote by \( C([0, T] \to B) \) the Banach space of all continuous maps \( u : [0, T] \to B \) with the norm

\[ \|u\| = \max_{0 \leq t \leq T} \|u(t)\|. \]

By symbol \( \widetilde{W}_2^1(\Omega) \) we denote the subspace of the Sobolev space \( W_2^1(\Omega) \) formed by functions whose trace is equal to \( \partial \Omega \) zero. Note that since \( \widetilde{W}_2^1(\Omega) \) is closed, the sum of a series of functions from \( \widetilde{W}_2^1(\Omega) \) converging in metric \( W_2^1(\Omega) \) also belongs to \( \widetilde{W}_2^1(\Omega) \).
Definition 1.1. By the solution of problem (1)-(4) we mean function \( u(x, y, t) \), represented in the form

\[
 u(x, y, t) = \frac{a-x}{a} \varphi(y) \mu(t) - v(x, y, t),
\]

where the function \( v(x, y, t) \) is a generalized solution from \( C([0, T] \to \tilde{W}^1_2(\Omega)) \) of the problem

\[
 v_t(x, y, t) - \Delta v(x, y, t) = -\frac{a-x}{a} \varphi''(y) \mu(t) + \frac{a-x}{a} \varphi(y) \mu'(t),
\]

with boundary conditions

\[
 v(x, y, t) \big|_{\partial\Omega} = 0,
\]

and initial condition

\[
 v(x, y, 0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b.
\]

Consequently,

\[
 v(x, y, t) = \int_0^t \sum_{n,m=1}^{\infty} e^{-\lambda_{nm}(t-s)} \frac{\varphi_m}{n} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \left( \frac{2\pi m^2}{b^2} \mu(s) + \frac{2}{\pi} \mu'(s) \right) ds,
\]

where \( \lambda_{nm} = (n\pi/a)^2 + (m\pi/b)^2 \), \( n, m = 1, 2, ... \) (see, e.g. [17], [18]).

Note that the class \( C([0, T] \to \tilde{W}^1_2(\Omega)) \) is a subset of the class \( W^1_2(\Omega) \) considered in the monograph [11] in order to define a problem with homogeneous boundary conditions. Thus, the generalized solution introduced above is also a generalized solution in the sense of monograph [11]. However, unlike a solution from the class \( W^1_2(\Omega) \), which is guaranteed to have a trace of almost all \( t \in [0, T] \), a solution from the class \( C([0, T] \to \tilde{W}^1_2(\Omega)) \) continuously depends on \( t \in [0, T] \) in the metric \( L^2_2(\Omega) \).

Proposition 1.1. Let \( \mu(t) \) be a smooth function on the half-line \( t \geq 0 \) and \( \varphi \in W^2_2[0, b] \). Then the function

\[
 u(x, y, t) = \int_0^t \mu(s) \sum_{n,m=1}^{\infty} \frac{2\pi n \varphi_m}{a^2} e^{-\lambda_{nm}(t-s)} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} ds,
\]

is the solution of the initial-boundary value problem (1)-(4).

Proof. A similar proof is given in the case [12]. We rewrite the solution to the problem in the form

\[
 u(x, y, t) = \frac{a-x}{a} \varphi(y) \mu(t) - \\
 - \int_0^t \sum_{n,m=1}^{\infty} e^{-\lambda_{nm}(t-s)} \frac{\varphi_m}{n} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \left( \frac{2\pi m^2}{b^2} \mu(s) + \frac{2}{\pi} \mu'(s) \right) ds.
\]
We show that function $v(x, y, t)$ belongs to class $C([0, T] \rightarrow \tilde{W}^1_2(\Omega))$. For this, it is enough to prove that the gradient of this function, taken in $(x, y) \in \Omega$, continuously depends on $t \in [0, T]$ in the norm of the space $L_2(\Omega)$. According to Parseval’s equality, the norm of this gradient is

$$\|\nabla v\|_{L^2(\Omega)}^2 = \sum_{n,m=1}^{\infty} \frac{|\varphi_m|^2}{n^2} \lambda_{nm} b_{nm}^2(t)$$

where

$$b_{nm}(t) = \int_0^t e^{-\lambda_{nm}(t-s)} \left( \frac{2\pi m^2}{b^2} \mu(s) + \frac{2}{\pi} \mu'(s) \right) ds.$$

From the Cauchy-Bunyakovsky inequality, we obtain the following estimate

$$b_{nm}(t) \leq C_1 \sqrt{\lambda_{nm}} + C_2 \frac{m^2}{\lambda_{nm}} \leq C_3 \frac{m}{\sqrt{\lambda_{nm}}}, \ t \geq 0.$$ 

From (6) and (7), we write

$$\varphi_m = \frac{2}{b} \int_0^b \varphi(y) \sin \frac{m\pi y}{b} dy = -\frac{2}{b} \varphi(y) \frac{b}{m\pi} \cos \frac{m\pi y}{b} \bigg|_{y=0} +$$

$$+ \frac{2}{m\pi} \int_0^b \varphi'(y) \cos \frac{m\pi y}{b} dy = \frac{2}{m\pi} \int_0^b \varphi'(y) \cos \frac{m\pi y}{b} dy = \frac{b}{m\pi} \varphi'_m. \quad (11)$$

Consequently,

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq C_3 \sum_{n,m=1}^{\infty} \frac{m^2 |\varphi_m|^2}{n^2} \leq C_3 \frac{\pi^2 b^2}{6} \sum_{m=1}^{\infty} m^2 \frac{|\varphi'_m|^2}{m^2} = C \|\varphi'\|_{L^2[0,b]}^2.$$ 

It is easy to see that with the equality (10) we can write the condition (8) of problem as the following

$$\theta = \frac{b/\alpha}{a/\beta} \int_0^t \int_0^b u(x, y, t) \ dx \ dy =$$

$$= \int_0^t \mu(s) ds \int_0^b \int_0^a \sum_{n,m=1}^{\infty} \frac{2\pi n}{a^2} \varphi_m e^{-\lambda_{nm}(t-s)} \ sin \frac{n\pi x}{a} \ sin \frac{m\pi y}{b} dx \ dy =$$

$$= \int_0^t \mu(s) \sum_{n,m=1}^{\infty} \frac{8b \varphi_m}{a m \pi} e^{-\lambda_{nm}(t-s)} \ sin^2 \frac{n\pi}{2\beta} \ sin^2 \frac{m\pi}{2\alpha} ds. \quad (12)$$
Set
\[ B(t) = \sum_{n,m=1}^{\infty} \rho_{nm} e^{-\lambda_{nm}t}, \]  
(13)
where \( \rho_{nm} \) defined by (9).

Then we get main integral equation
\[ \int_0^t B(t-s)\mu(s)ds = \theta, \ t > 0. \]  
(14)

**Proposition 1.2.** For \( B(t) \) defined by (13) the following estimate
\[ 0 < B(t) \leq C \sqrt{t}, \ 0 < t \leq 1, \]  
(15)
is valid.

**Proof.** From (9) and (11), we may write
\[ 0 < \rho_{nm} \leq C_0 \varphi'_m \frac{m^2}{e^{s^2 / a^2}}. \]
Hence, using the definition (13) we get
\[ 0 < B(t) \leq C_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \varphi'_m \frac{m^2}{e^{s^2 / a^2}}. \]  
(16)

Set
\[ K(t) = \sum_{m=1}^{\infty} \varphi'_m \frac{e^{-s^2 / a^2}}{t} \]  
with \( t > 0 \).

This function for any \( T > 0 \) and \( 0 \leq t \leq T \) satisfies inequalities
\[ 0 < K(T) \leq K(t) \leq K(0). \]  
(17)

For any \( p > 0 \) consider the following relations:
\[ \sum_{n=1}^{\infty} e^{-pn^2} = \sum_{n=1}^{\infty} \int_{s=1}^{n+1} e^{-p(s)^2} ds = \int_{s=1}^{\infty} e^{-p(s)^2} ds = \int_{s=1}^{\infty} e^{-ps^2} e^{p(s^2-|s|^2)} ds, \]
where \(|s|\) is integer part of \( s \).

Note that \( e^{p(s^2-|s|^2)} = e^{p(s^2-|s|^2)+|s|^2} \leq e^{2ps^2} \). Then we obtain
\[ \int_{s=1}^{\infty} e^{-ps^2} e^{p(s^2-|s|^2)} ds \leq \int_{s=1}^{\infty} e^{-ps^2+2ps^2} ds = -\int_{s=1}^{\infty} e^{p(s^2-|s|^2)} ds. \]
Hence, for \( 0 < p \leq \text{const} \) we get
\[ \sum_{n=1}^{\infty} e^{-pn^2} \leq \int_{s=1}^{\infty} e^{-ps^2} e^{p(s^2-|s|^2)} ds \leq e^p \int_{s=1}^{\infty} e^{-ps^2} ds \leq C \sqrt{p}. \]  
(18)

Put \( p = \frac{\pi^2 t}{a^2} \). Then required estimate (15) follows from (16)-(18).
2 Estimate of Minimal Time

We consider the following integral equation

\[ \int_{0}^{t} B(t-s) \mu(s) ds = \theta, \quad t \geq T, \]  

(19)

where

\[ B(t) = \sum_{n,m=1}^{\infty} \rho_{nm} e^{-\lambda_{nm} t}. \]  

(20)

**Proposition 2.1.** For the function defined by equality (20) the following estimate

\[ B(t) \geq \rho_{11} e^{-[(\pi/a)^2+(\pi/b)^2]t}, \]  

(21)

is valid.

**Proof.** The proof comes from functional series defined by (20) is non-negative.

We introduce a specific heating as

\[ Q(t) = \int_{0}^{t} B(t-s) ds = \int_{0}^{t} B(s) ds = \sum_{n,m=1}^{\infty} \frac{\rho_{nm}}{\lambda_{nm}} \left( 1 - e^{-\lambda_{nm} t} \right). \]  

(22)

The physical meaning of this function is evident: \( Q(t) \) equals the average temperature of \( \Omega \) in case where the heater is acting unit load (see, e.g. [1], [2]).

It is clear that \( Q(0) = 0 \) and \( Q'(t) = B(t) \geq 0 \).

Set

\[ Q^* = \lim_{t \to \infty} Q(t) = \int_{0}^{\infty} B(s) ds. \]  

(23)

Obviously, the average temperature of \( \Omega \) in the case where the heater is acting with unit load cannot exceed \( Q^* \).

**Proposition 2.2.** Let

\[ 0 < \theta < MQ^*. \]  

(24)

Then there exist \( T > 0 \) and a real-valued measurable function \( \mu(t) \) so that \( |\mu(t)| \leq M \) and the following equality

\[ \int_{0}^{T} B(T-s)\mu(s) ds = \theta \]  

(25)

is valid.
Proof. This follows from the properties of the function $Q$. Indeed, if we set $\mu(t) = M$ then
\[
\int_0^t B(t - s)\mu(s)\,ds = M \int_0^t B(t - s)\,ds = MQ(t),
\]
and because of (25) there exists $T > 0$ so that $MQ(T) = \theta$.

Remark 2.1. It is clear that the value $T$, which was found in Proposition 2.2, gives a solution to the problem. Namely, $T$ is the root of the equation
\[
Q(T) = \frac{\theta}{M}.
\] (26)

However, the main idea of the present work is to establish an acceptable estimate for the value of the minimal time $T$ (see, e.g. [3]).

Proposition 2.3. Let
\[
0 < \theta < \frac{\rho_{11} M a^2 b^2}{\pi^2 (a^2 + b^2)}.
\] (27)

Then there exists $T > 0$ so that
\[
T < -\frac{a^2 b^2}{\pi^2 (a^2 + b^2)} \ln \left(1 - \frac{\theta \pi^2 (a^2 + b^2)}{\rho_{11} M a^2 b^2}\right)
\] (28)

and the equality (26) is fulfilled.

Proof. For obtaining the required estimate we use Proposition 2.1. We may write
\[
Q(t) = \int_0^t B(s)\,ds \geq \rho_{11} \int_0^t e^{-[(\pi/a)^2 + (\pi/b)^2]s}\,ds =
\]
\[
= \frac{\rho_{11} a^2 b^2}{\pi^2 (a^2 + b^2)} \left(1 - e^{-[(\pi/a)^2 + (\pi/b)^2]t}\right)
\] (29)

Consider the following equation for the defining of $T_0$:
\[
\frac{\rho_{11} a^2 b^2}{\pi^2 (a^2 + b^2)} \left(1 - e^{-[(\pi/a)^2 + (\pi/b)^2]T_0}\right) = \frac{\theta}{M}.
\] (30)

Then
\[
T_0 = -\frac{a^2 b^2}{\pi^2 (a^2 + b^2)} \ln \left(1 - \frac{\theta \pi^2 (a^2 + b^2)}{\rho_{11} M a^2 b^2}\right).
\]

In accordance with (29) and (30) we may write
\[
0 < \frac{\theta}{M} \leq Q(T_0).
\]

Then obviously there exists $T$, $0 < T < T_0$, which is a solution to the equation (26).
Proposition 2.4. Let $T > 0$ satisfies the equality (26) and condition (27).
Then there exist $T_1 > T$ and a measurable real-valued function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality

$$\int_0^{b/a} \int_0^{a/\beta} u(x,y,t) \, dx \, dy = \theta, \quad T \leq t \leq T_1,$$

is valid.

Proof. According to the following

$$\int_0^t B(t-s)\mu(s)ds = \theta, \quad 0 \leq t \leq T_1, \quad (31)$$

it is enough to prove that there exists solution of the equation

$$\int_0^t B(t-s)\mu(s)ds = f(t), \quad 0 \leq t \leq T_1, \quad (32)$$

where

$$f(t) = \begin{cases} MQ(t), & \text{if } 0 \leq t \leq T, \\ \theta, & \text{if } T < t \leq T_1. \end{cases} \quad (33)$$

The solution (33) is piecewise smooth and, according to equality (26), is continuous.

Set

$$\mu(t) = \begin{cases} M, & \text{if } 0 \leq t \leq T, \\ \mu_1(t), & \text{if } T < t \leq T_1, \end{cases} \quad (34)$$

where $\mu_1(t)$ is the solution of the following integral equation

$$\int_0^T B(t-s)Mds + \int_T^t B(t-s)\mu_1(s)ds = \theta, \quad T \leq t \leq T_1. \quad (35)$$

After differentiating this equation we get

$$B(0)\mu_1(t) + \int_T^t B'(t-s)\mu_1(s)ds = M[B(t-T) - B(t)]. \quad (36)$$

According to (20) $B(0)$ positive and $B(t)$ function is convergence function on given interval. Hence, equation (36) has a unique solution $\mu_1(t)$ for all $t \geq T$, which is continuous function on the half line $t \geq T$. Besides,

$$\mu_1(T) = M \left(1 - \frac{B(T)}{B(0)}\right) < M,$$
and there exists $T_1 > T$ so that

$$|\mu_1(t)| \leq M, \quad T \leq t \leq T_1.$$  

It is clear that this function is the unique solution of the equation (35). Hence, the function (34) is piecewise continuous and satisfies equation (32). Consequently, this function $\mu(t)$, which has a jump at the point $t = T$, is the required solution.

The proof of Theorem 0.1 follows now easily from Proposition 2.3 and Proposition 2.4.

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References


