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RESTORING THE FUNCTION SET BY INTEGRALS FOR THE FAMILY OF PARABOLAS ON THE PLANE

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Abstract

Integral geometry is one of the most important sections of the theory of ill-posed problems of mathematical physics and analysis. The urgency of the problems of integral geometry is due to the development of tomographic methods, which raise the requirements for the depth of the applied results, the fact that the solution of problems of integral geometry reduces a number of multidimensional inverse problems for partial differential problems, as well as the internal development needs of the theory of ill-posed problems of mathematical physics and analysis. In this work we consider the problem of reconstructing a function from a family of parabolas in the upper half-plane with a weight function of a new kind. The uniqueness and existence theorems of the solution of the problem are proved and the inversion formula is derived. It is shown that the solution of the problem posed is weakly ill-posed, that is, stability estimates are obtained in spaces of finite smoothness.

Keywords: Problems of integral geometry, weakly ill-posed problems, Fourier transforms, theorem of uniqueness and existence, weight function, finite function.


Introduction

The problems of integral geometry naturally arise in the study of many mathematical models in such areas of wide application, like seismic, interpretation of geophysical and aerospace observations, various processes described by kinetic equations and others. The methods developed here is the mathematical basis of computational tomography, rapidly developing area of modern science. One of the central problems of integral geometry is the restoration of a function if its integrals over given manifolds are known.

We give a general statement of the problem of integral geometry [1].

Let \( u(x) \) - be a sufficiently smooth function defined in \( \mathbb{R}^n \) and \( \{S(y)\} \) – is a family of piecewise smooth manifolds in this space depending on the parameter \( y = (y_1, y_2, \ldots y_n) \).

Let the integrals of the function \( u(x) \) be defined es

\[
\int_{S(y)} g(x,y)u(x)ds = v(\lambda),
\]

(1)
where \( g(x, y) \) is a given weight function, \( ds \) is the measure element on \( S(y) \). It is required by function \( v(\lambda) \) to restore the function \( u(x) \).

The main questions that arise in the study of this problem are as follows. The first and most fundamental question is whether a function definition defines a function uniquely? The first and most fundamental question, does the definition of the function \( v(\lambda) \) uniquely determine the function \( u(x) \). Further, how to find a function \( u(x) \) by function \( v(\lambda) \)? The question is important: how to get an analytical formula expressing \( u(x) \) in terms of \( v(\lambda) \)? It should be noted here that this is not possible in all cases. And finally, the question is naturally related to the existence theorem of the solution to the problem: What are the necessary and sufficient conditions for belonging to the class of functions representable via integral (1)?


Sufficiently general results on the uniqueness and stability of solving problems of integral geometry in the case where the manifold on which the integration is carried out have the kind of paraboloids, weight functions and diversity are invariant under the group of all the movements along a fixed hyperplane were obtained by V.G.Romanov [3].

The uniqueness of the solution of significantly wider classes of integral geometry problems in the band, considered as weakly ill-posed, was established by V.G.Romanov [4]. In the works of A.L. Buchheim [5, 6] receiving formulas were obtained for the problem of recovering a function through integrals from it over paraboloids in a half space \( y > 0 \), and the formula given in [5] contains only a finite number of derivatives of the data. In [6] using the technique of scales of Banach spaces, a uniqueness theorem was proved for the solution of the integral geometry problem in a band on parabolas with a weight function that is analytic in some variables.

In the works of A.Kh. Begmatov [7] is studying a new class of problems of the inverse of the ray transform with incomplete given. By the characteristics of instability, this is a very ill-posed problem. The results obtained in these articles had a beneficial effect on the work [8, 9, 10, 11, 12, 13, 14].

Weakly ill-posed integral geometry problems Voltaire type with weight functions which are particularly studied in [15].

Uniqueness theorems, stability estimates, and inversion formulas for weakly ill-posed problems of integral geometry over special curves and surfaces with singularities of vertices were obtained in [16].

In the works [17, 18, 19, 20, 21], new classes of the problem of integral geometry were studied and new approaches to the study of problems of recovering functions from weighted functions with a singularity were introduced.

As is known, the problem of integral geometry and the inverse problem have a close connection. A well-known monograph by M.M. Lavrent’ev [22]. The questions of weakly generalized solvability of a nonlinear inverse problem studied in work [23].

In this paper, we consider the problem of reconstructing a function from a family of parabolas in the upper half-plane with a weight function of a new kind. The
uniqueness and existence theorems of the solution of the problem are proved and the inversion formula is derived. It is shown that the solution of the problem posed is weakly ill-posed, that is, stability estimates are obtained in spaces of finite smoothness.

1 Formulation of the problem

We introduce the notation that will be used below:

$$(x, y) \in \mathbb{R}^2, (\xi, \eta) \in \mathbb{R}^2, \Omega = \{(x, y): x \in \mathbb{R}^1, y \in [0, l]\},$$

here $0 < l < \infty$.

In strip $\Omega$, we consider a family of $P(x, y)$ curves, which are determined by the relations

$$P(x, y) = \{(\xi, \eta): y - \eta = (x - \xi)^2, \, 0 \leq \eta \leq y\}.$$

Statement of Problem 1. Restore the function of two variables $u(x, y)$, if in the strip $\Omega$ the integrals from it over the curves of the family $P(x, y)$ with the weight function $g(x, \xi)$ are known:

$$\int_{x - \sqrt{y}}^{x + \sqrt{y}} g(x, \xi) u(\xi, y - (x - \xi)^2) d\xi = f(x, y). \tag{2}$$

Problem 1 is a problem of integral geometry of Volterra type [15, 17].

Such problems on linear manifolds and other clearly defined curves and surfaces have numerous applications in computer, seismic and ultrasound tomography.

Function $u(x, y)$ is a function from class $U$, which have all continuous partial derivatives up to the eighth order inclusive and are compactly supported with support $\Omega$. For definiteness, we have

$$\text{supp } u \subset D = \{(x, y): -a < x < a, \, 0 < y < l\}, \, 0 < a < \infty.$$

We define on the right hand side of (2) equation for $y < 0$.

We introduce the function

$$f^*(x, y) = \begin{cases} f(x, y), & \text{for } y \geq 0, \\ 0, & \text{for } y < 0. \end{cases}$$

As follows from the statement of problem and the conditions imposed on the function $u(x, y)$, the Fourier transform with respect to the function $f^*(x, y)$ can be applied $y$.

$$\hat{f}(\lambda, \mu) = \int_{-\infty}^{\infty} e^{-i\mu y} \hat{f}^*(\lambda, y) dy = \int_{0}^{\infty} e^{-i\mu y} \hat{f}(\lambda, y) dy.$$
We introduce the following functions

\[
I(\lambda, \mu) = \int_0^\infty e^{-i\mu \tau} \cos(\lambda \sqrt{\tau}) d\tau, \tag{3}
\]

\[
I_1(\lambda, y - \eta) = \int_0^\infty \frac{d\mu}{(1 + \mu^4) I(\lambda, \mu)} \tag{4}
\]

\[
I_2(x - \xi, y - \eta) = \int_{-\infty}^{+\infty} e^{i\lambda(x-\xi)} I_1(\lambda, y - \eta) \frac{1 + \mu^4}{(1 + \lambda^4)} d\lambda. \tag{5}
\]

### 1.1 Uniqueness Theorem

The following theorem holds:

**Theorem 1.** Let the function \( f(x, y) \) be known for all \( (x, y) \in \Omega \), the weight function has the form:

\[
g(x, \xi) = |x - \xi|.
\]

Then the solution of problem 1 in the class of eight times continuously differentiable compactly supported functions in the strip \( \Omega \)

\[
u(x, y) = \int_0^\infty \int_{-\infty}^{\infty} I_2(x - \xi, y - \eta)(1 + \partial_4 \partial_4)(1 + \partial_\eta^4) f(\xi, \eta) d\xi d\eta, \tag{6}
\]

and the inequality holds

\[
\| u \|_{L^2(\Omega)} \leq C \| f \|_{W_4^4(\Omega)},
\]

where \( C \) is some constant.

**Proof.** We rewrite substitution the equation (2) to a more convenient form:

\[
\frac{1}{2} \int_0^y \left[ u(x - h, \eta) + u(x + h, \eta) \right] d\eta = f(x, y), \tag{7}
\]

where \( h = \sqrt{y - \eta} \).

We apply the Fourier transform in the first variable to both sides of equation (7). We obtain:

\[
\int_0^y \cos \lambda h \hat{u}(\lambda, \eta) d\eta = \hat{f}(\lambda, y). \tag{8}
\]

We apply one-sided Fourier transform with respect to \( y \) to the equation (8):

\[
\hat{f}(\lambda, \mu) = \int_0^\infty e^{-i\mu y} \int_{0}^{y} \cos \lambda h \hat{u}(\lambda, \eta) d\eta dy = \int_0^\infty \hat{u}(\lambda, \eta) \int_{\eta}^{\infty} e^{-i\mu y} \cos \lambda \sqrt{y - \eta} dy d\eta.
\]
Having made a substitution in this equality $\tau = y - \eta$, we obtain:

$$\hat{f}(\lambda, \mu) = \hat{u}(\lambda, \mu) \int_{0}^{\infty} e^{-i\mu \tau} \cos \lambda \sqrt{\tau} d\tau.$$ (9)

We need to estimate the function $I(\lambda, \mu)$, from below, which allows us to show the separation of the desired function $u(x, y)$ from zero.

Having made a substitution $\sqrt{\tau} = t$ and taking into account $d\tau = 2t dt$, in the integral (3), we have:

$$I = 2 \int_{0}^{\infty} e^{-i\mu t} \cos \lambda t dt = 2 \int_{0}^{\infty} t \cos \mu t \cos \lambda t dt - 2i \int_{0}^{\infty} t \sin \mu t \cos \lambda t dt.$$

The last integral is easy to calculate (see. [24]):

$$I = -\frac{1}{\mu} \left( i - \frac{\lambda}{\sqrt{\mu}} \int_{0}^{\frac{2\sqrt{\pi}}{\lambda}} e^{-i\tau^{2}} d\tau \right).$$

The function module $I(\lambda, \mu)$ has the form:

$$|I| = \left| -\frac{1}{\mu} \left( i - \frac{\lambda}{\sqrt{\mu}} \int_{0}^{\frac{2\sqrt{\pi}}{\lambda}} e^{-i\tau^{2}} d\tau \right) \right| \geq \left| \frac{1}{\mu} \left( 1 - \frac{\lambda}{\sqrt{\mu}} \int_{0}^{\frac{2\sqrt{\pi}}{\lambda}} e^{-i\tau^{2}} d\tau \right) \right|.$$ (11)

We show the following inequality

$$\left| 1 - \frac{\lambda}{\sqrt{\mu}} \int_{0}^{\frac{2\sqrt{\pi}}{\lambda}} e^{-i\tau^{2}} d\tau \right| > \frac{1}{2}. \quad (10)$$

From the Fresnel integrals, it is easy to verify the following inequality $|1 - \frac{\lambda}{2\sqrt{\pi}}| > \frac{1}{2}$, where

$$\frac{\lambda}{2\sqrt{\mu}} \in \left( -\infty, \frac{1}{2\sqrt{\pi}} \right) \cup \left( \frac{3}{2\sqrt{\pi}}, +\infty \right).$$

For others $\frac{\lambda}{2\sqrt{\mu}}$ we can obtain that inequality (10) is also valid for $\frac{\lambda}{2\sqrt{\mu}} \in \left[ \frac{1}{2\sqrt{\pi}}; \frac{3}{2\sqrt{\pi}} \right]$ using the table of Fennel integrals.

Thus,

$$|I| = \left| 2 \int_{0}^{\infty} e^{-i\mu t} \cos \lambda t dt \right| \geq \frac{1}{2\mu}. \quad (11)$$

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From equation (9), and assuming (11), we obtain
\[
\hat{u}(\lambda, \mu) = \frac{1}{I(\lambda, \mu)} \hat{f}(\lambda, \mu).
\] (12)

Divide and multiply the right side of equality (12) by \((1 + \mu^4)\):
\[
\hat{u}(\lambda, \mu) = \frac{1}{(1 + \mu^4)I(\lambda, \mu)} (1 + \mu^4) \hat{f}(\lambda, \mu).
\] (13)

On the right side of equation (13), the function \(\hat{f}(\lambda, \mu)\) takes the form
\[
\hat{f}(\lambda, \mu) = \int_0^{+\infty} e^{-i\mu\eta} \hat{f}(\lambda, \eta) d\eta.
\]

Applying integration by parts four times, considering the property of differentiating the Fourier transform, we obtain the following
\[
\int_0^{+\infty} e^{-i\mu\eta} \hat{f}(\lambda, \eta) d\eta = \frac{1}{\mu^4} \int_0^{+\infty} e^{-i\mu\eta} \frac{\partial^4 \hat{f}(\lambda, \eta)}{\partial \eta^4} d\eta.
\] (14)

Substituting (14) on (13), we obtain
\[
\hat{u}(\lambda, \mu) = \frac{1}{(1 + \mu^4)I(\lambda, \mu)} \int_0^{+\infty} e^{-i\mu\eta} \left( E + \frac{\partial^4 \hat{f}(\lambda, \eta)}{\partial \eta^4} \right) \hat{f}(\lambda, \eta) d\eta.
\] (15)

We apply the inverse Fourier transform in variable \(\mu\) to equation (15).
\[
\hat{u}(\lambda, y) = \int_0^{+\infty} I_1(\lambda, y - \eta) \left( 1 + \frac{\partial^4 \hat{f}(\lambda, \eta)}{\partial \eta^4} \right) \hat{f}(\lambda, \eta) d\eta.
\] (16)

Divide and multiply the right side of equality (16) by \((1 + \lambda^4)\):
\[
\hat{u}(\lambda, y) = \int_0^{+\infty} \frac{I_1(\lambda, y - \eta)}{(1 + \lambda^4)} \left( 1 + \frac{\partial^4 \hat{f}(\lambda, \eta)}{\partial \eta^4} \right) \hat{f}(\lambda, \eta) d\eta.
\] (17)

In the integral
\[
\hat{f}(\lambda, \eta) = \int_{-\infty}^{\infty} f(\xi, \eta) e^{-i\lambda \xi} d\xi,
\]
apply integration by parts four times, and apply the property of differentiating the Fourier transform, we obtain the following
\[
\int_{-\infty}^{\infty} f(\xi, \eta) e^{-i\lambda \xi} d\xi = \frac{1}{\lambda^4} \int_{-\infty}^{\infty} \frac{\partial^4 f(\xi, \eta)}{\partial \xi^4} e^{-i\lambda \xi} d\xi.
\] (18)
From here (17) takes the form
\[ \hat{u}(\lambda, y) = \int_0^\infty \int_{-\infty}^{\infty} e^{-i\lambda \xi} \left( 1 + \frac{\partial^4}{\partial \eta^4} \right) \left( 1 + \frac{\partial^4}{\partial \xi^4} \right) f(\xi, \eta) d\xi d\eta. \] (19)

Applying the inverse Fourier transform with respect to \( \lambda \) to equation (19) and using the convolution theorem, as well as the differentiation property of the Fourier transform, we obtain:
\[ u(x, y) = \int_0^{+\infty} \int_{-\infty}^{+\infty} e^{i\lambda(x-\xi)} I_1(\lambda, y-\eta) \left( 1 + \frac{\partial^4}{\partial \eta^4} \right) \left( 1 + \frac{\partial^4}{\partial \xi^4} \right) f(\xi, \eta) d\xi d\eta. \]

It follows from (4) that the function \( I_1(\lambda, y-\eta)(1+\lambda^4)^{-1} \) is the Fourier transform with respect to the first variable of the function
\[ I_2(x-\xi, y-\eta) = \int_{-\infty}^{+\infty} \frac{e^{i\lambda(x-\xi)} I_1(\lambda, y-\eta)}{(1 + \lambda^4)} d\lambda. \]

Meaning
\[ u(x, y) = \int_0^{+\infty} \int_{-\infty}^{+\infty} I_2(x-\xi, y-\eta) \left( 1 + \frac{\partial^4}{\partial \eta^4} \right) \left( 1 + \frac{\partial^4}{\partial \xi^4} \right) f(\xi, \eta) d\xi d\eta. \] (20)

Formula (20) has a local character in the variable \( y \). Given the condition \( \text{supp } u \subset \Omega \), it is clear that representation (20) for solving equation (12) also holds for \( l < \infty \). Then from (8), (12) and (20) follows the uniqueness of the solution of the original problem I in the class of functions \( C_0^2(\Omega) \).

Notice, that
\[ ||u||_{L_2(\Omega)} = \int_0^{+\infty} \int_{-\infty}^{+\infty} |u(x, y)|^2 dxdy \leq ||I_2||_{L_2(\Omega)} ||f||_{W_4^4(\Omega)}. \]

From equation (8), taking into account (12), it is easy to obtain the following inequality
\[ |\hat{u}(\lambda, \mu)| \leq \frac{1}{|I(\lambda, \mu)|} |\hat{f}(\lambda, \mu)|. \]

Where we deduce the estimate
\[ \int_0^{+\infty} \int_{-\infty}^{+\infty} |\hat{u}(\lambda, \mu)|^2 d\lambda d\mu \leq \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{|I|^2} |\hat{f}(\lambda, \mu)|^2 d\lambda d\mu \] (21)
Using the properties of transform Fourier differentiation, triangle inequality of norms, and taking (20) and (21), and the conditions imposed on the function $u$ obtain the estimate

$$||u(x,y)||_{L^2(\Omega)} \leq C||f||_{W^{4,4}(\Omega)},$$

where $C$ - is some constant.

From which follows the uniqueness of the solution of problem 1. □

References


