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OSCILLATORY INTEGRALS AND WEIERSTRASS POLYNOMIALS

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Abstract

In this work we consider some applications of the Weierstrass preparation theorem and Weierstrass pseudopolynomials to study of behavior of the oscillatory integrals and Fourier transforms with analytic and smooth phases with critical points.

**Keywords:** oscillatory integral, Fourier transform, Randoll maximal function, Weierstrass preparation theorem, pseudoalgebraic set.

**Mathematics Subject Classification (2010):** 42A38, 32D10, 31C45.

Introduction

An oscillatory integral is called to be the integral of the form:

\[
J(\lambda, \sigma) := \int_{\mathbb{R}^n} a(x, \sigma) e^{i\lambda\Phi(x, \sigma)} \, dx,
\]

where \(a \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^m)\) is called to be an amplitude function and \(\Phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)\) is a smooth real-valued function co-called a phase function and \(\lambda\) is a real parameter. It is required to study the asymptotic behavior of the oscillatory integral (1) as \(\lambda \to \infty\).

Actually, by using standard partition of unity arguments we may suppose that \(a\) is concentrated in a sufficiently small neighborhood of a fixed point say in a small neighborhood \(U \times V\) of the origin, \(\text{supp}(a) \subset U \times V \subset \mathbb{R}^n \times \mathbb{R}^m\).

It should be noted that behavior of the oscillatory integrals with analytic phases and also smooth phases with critical points having finite multiplicity related to problems of multidimensional analytic functions theory.

Oscillatory integrals play an essential role in many branches of mathematics. Especially some problems from mathematical physics, theory of probability and analytic number theory lead to investigate oscillatory integrals connected to Fourier transform of distributions, averaging operators, damped oscillatory integrals etc. In particular, Fourier transform of "delta" distributions associated to surfaces in Euclidean spaces is a special case of the oscillatory integrals (see [1], [2], [7], [10], [11], [12]).

In this talk we consider some applications of the Weierstrass theorem and Weierstrass polynomials, also new versions of related results to estimate the oscillatory integrals related to the Fourier transform of surface-carried measures. First, we formulate a series of standard problems from the theory of oscillatory integrals, which we are going discuss with you.
Let $S \subset \mathbb{R}^{n+1}$ be a smooth hypersurface and $\psi \in C_0^\infty(\mathbb{R}^{n+1})$. Consider the oscillatory integral, which is given by Fourier transform of the measure $d\mu(x)$:

$$d\hat{\mu}(\xi) := \int_S e^{i\langle x, \xi \rangle} d\mu(x),$$

where $d\mu(x) := \psi(x)dS$ with an amplitude function $0 \leq \psi \in C_0^\infty(S)$, $dS$ is the induced Lebesgue measure on the surface $S$ and $(x, \xi)$ is the scalar product of the vectors $x$ and $\xi$. If $S(\sigma) \subset \mathbb{R}^{n+1}$ is a family of smooth hypersurfaces depending on the parameters $\sigma \in \mathbb{R}^m$ then by the analogy we can define $d\mu_\sigma$ and $d\hat{\mu}_\sigma(\xi)$ respectively.

The main task is to study the behavior of oscillatory integrals in the neighborhood of infinity, e.g. when $|\xi|$ gets large.

**Problem 1.** Find the greatest lower bound $p_0$ of the set $\{p : d\hat{\mu} \in L^p(\mathbb{R}^{n+1})\},$ $p_0 = \inf\{p : d\hat{\mu} \in L^p(\mathbb{R}^{n+1})\}$.

The next problem is connected to the classical Randoll maximal function. Let’s define the Randoll maximal function:

$$M(\omega) := \sup_{r>0} r^{\frac{n}{2}} |d\hat{\mu}(r\omega)|,$$

where $\omega \in \mathbb{R}^{n+1}$. Note that by classical Card Theorem for almost every $\omega \in \mathbb{R}^{n+1}\setminus\{0\}$ the maximal function $M(\omega)$ is a finite positive number.

**Remark** Note that $M(\omega)$ is a positive homogeneous function of degree $-n/2$, $M(\tau \omega) = \tau^{-n} M(\omega)$. So that, it is enough to study restricting it on $\omega \in \Sigma^n$, where $\Sigma^n$ is the unite sphere centered at the origin.

**Problem 2.** Find the smallest upper bound $p_0$ of the set $\{p : M \in L^p(\Sigma^n)\},$ $p_0 = \sup\{p : M \in L^p(\Sigma)\}$.

In 1976, Stein E.M. (see [18]) proved that, for the Euclidean unit sphere $S = S^n \subset \mathbb{R}^{n+1}, n \geq 1$, the $\hat{\mu}(\xi)$ is bounded on $L^p(\mathbb{R}^{n+1})$ if and only if $p > (n + 1)/n$. The key property of spheres which allows to prove such results is the non-vanishing of the Gaussian curvature on spheres and the following transversality assumption:

**Assumption** The affine tangent plane $x + T_x S$ to $S$ in $x$ does not pass through the origin $0 \in \mathbb{R}^{n+1}$ for every $x \in S$.

These results became the starting point for intensive studies of various classes of operators $\hat{\mu}(\xi)$ associated to smooth hypersurfaces under the transversality assumption and non-vanishing of the Gaussian curvature $K(x)$. C.D. Sogge and E.M. Stein [17] prove that if the Gaussian curvature $K(x) \neq 0$, then

$$\{\hat{\mu}(\xi) = O(|\xi|^{-n/2})(as |\xi| \to +\infty)\}. $$

However, in case of $K(x) = 0$ in some point of $S$, the (3), generally speaking, does not holds. In this reason, by C.D. Sogge and E.M. Stein [17] were introduced the following damped oscillatory integrals:

$$\hat{\mu}_q(\xi) := \int_S e^{i\langle \xi, x \rangle} |K(x)|^q \psi(x) dS(x).$$
They proved that if \( q \geq 2n \), then the integral (4) decays as \( O(|\xi|^{-n/2}) \) (as \( |\xi| \to +\infty \)).

**Problem 3.** Find the greatest lower bound \( q_0 \) of the set

\[
\{ q : \hat{\mu}_q(\xi) = O(|\xi|^{-n/2})(as |\xi| \to +\infty) \}, \quad q_0 = \inf \{ q : \{\mu}_q(\xi) = O(|\xi|^{-n/2})(as |\xi| \to +\infty) \}.
\]

This problem for a family of analytic hypersurfaces had been considered in the paper [4].

As noted before the formulated above problems for the analytic hypersurfaces closely related to the Weierstrass representations. In the next section we give some results in this directions, which are then in the section 3 applied to the problems listed above.

1 **Weierstrass representation**

1. The well-known Weierstrass theorem states that if \( f(z, w) \) is a holomorphic function at a point \( (z_0, w_0) \in \mathbb{C}_z \times \mathbb{C}_w \) and \( f(z_0, w_0) = 0 \), \( f(z_0, w) \neq 0 \) then in some neighborhood \( U = V \times W \) of this point \( f \) is represented as

\[
f(z, w) = [(w - w_0)^m + c_{m-1}(z)(w - w_0)^{m-1} + \cdots + c_0(z)] \varphi(z, w),
\]

where \( m \geq 1 \) is the order of zero of \( f(z_0, w) \) at \( w = w_0, c_k(z), k = 0, 1, ..., m - 1, \) are holomorphic in \( V \), \( c_k(z_0) = 0 \) and \( \varphi(z, w) \) is holomorphic in \( U, \varphi(z, w) \neq 0, (z, w) \in U \).

Pseudopolynomial

\[
(w - w_0)^m + c_{m-1}(z)(w - w_0)^{m-1} + \cdots + c_0(z)
\]

is called Weierstrass polynomial.

In the considering below results the phase function is an analytic function at a fixed critical point, requiring a condition \( f(z_0, w) \neq 0 \).

It is natural to expect the validity of Weierstrass theorem without requiring a condition \( f(z_0, w) \neq 0 \), in the following form, that in some neighborhood \( U = V \times W \) of \( (z_0, w_0) \) the function is represented as

\[
f(z, w) = [c_m(z)(w - w_0)^m + c_{m-1}(z)(w - w_0)^{m-1} + \cdots + c_0(z)] \varphi(z, w).
\]

where \( c_k(z), k = 0, 1, \ldots, m \) are holomorphic in \( V \) and \( \varphi \) is holomorphic in \( U, \varphi(z, w) \neq 0, \forall (z, w) \in U \). Such kind of results will be useful to studying of the oscillatory integrals and in estimates for maximal operators.

When \( n = 1 \) representation (6) takes place, because in this case it is easy to prove that in a neighborhood \( U = V \times W \) of \( (z_0, w_0) \) the function \( f(z, w) \) is represented as \( f(z, w) = (z - z_0)^j \varphi(z, w) \), where \( j \geq 0, \varphi(z, w) \in \mathcal{O}(U), \varphi(z_0, w) \neq 0 \). However,
the well-known Osgood counterexample [6] shows that when \( n > 1 \) it is not always possible to decompose the function into factors (6). We reduce this example.

**Osgood’s example.** We fix a specific transcendental function \( \psi(\xi) = \xi + \sum_{k=2}^{\infty} \xi^k \) in the unit ball \( |\xi| < 1 \) and put \( q(\xi) = \xi + \psi(\xi) \). In some neighborhood of zero \( w = q(\xi) \) has the inverse \( \xi = q^{-1}(w) \). Set \( f(z_1, z_2, w) = z_1q^{-1}(w) - z_2 \). Then \( f \) is holomorphic in some neighborhood \((0, 0, 0)\) and \( f(0, 0, w) = 0 \). We note, that each point of the circle \( |\xi| = 1 \) is singular for the function \( \psi(\xi) \), i.e. the function \( \psi(\xi) \) cannot analytically continue to any neighborhood \( U, U \cap \{|\xi| = 1\} \neq \emptyset \). If for \( f \) the Weierstrass representation (6) holds, then \( f(z_1, z_2, w) = 0 \) has a pseudoalgebraic solution \( c_m(z)w^m + c_{m-1}(z)w^{m-1} + \cdots + c_0(z) = 0 \), but its solution \( w = q\left(\frac{z_2}{z_1}\right) \) cannot lie in the analytic set \( c_m(z)w^m + c_{m-1}(z)w^{m-1} + \cdots + c_0(z) = 0 \), since the cone \(|\frac{z_2}{z_1}| = 1\) is a singular set for the function \( q\left(\frac{z_2}{z_1}\right) \).

There is a global multidimensional (in \( w \)) option (6) for arbitrary \( f(z, w) \in \mathcal{O}(D_z \times \mathbb{C}_w) \). [Accepted for publication to Annales Polonici Mathematici]

**Theorem 1.** Let \( f(z, w) \) be holomorphic in a domain \( \Omega = D_z \times \mathbb{C}_w \subset \mathbb{C}^{n+1} \), where any second Cousin problem is solvable in a domain \( D \subset \mathbb{C}^n \). Let us denote by \( n_f(z_0) \) the number of zeros of the entire function \( f(z_0, w) \) of variable \( w \in \mathbb{C} \) taking into account multiplicity. We put \( n_f(z_0) = -1 \) if \( f(z_0, w) \equiv 0 \). If the set \( \mathfrak{S}(f) := \{ z \in D : n_f(z) < \infty \} \) is not pluripolar in \( D \), then the function \( f(z, w) \) is represented as (6), where \( c_k(z) \in \mathcal{O}(D), k = 0, 1, \ldots, m, \varphi(z, w) \in \mathcal{O}(\Omega), \varphi(z, w) \neq 0 \) for any \( (z, w) \in \Omega \).

We give a multidimensional (in \( w \)) analogue of this theorem in the following form. The pseudopolynomial of degree \( m \) in a domain \( \Omega = D_z \times \mathbb{C}_w \subset \mathbb{C}^{n+k} \) is an expression in the form:

\[
Q_m(z, w) = \sum_{|\alpha|=m} C_\alpha(z)w^\alpha + \sum_{|\alpha|=m-1} C_\alpha(z)w^\alpha + \cdots + C_0(z),
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \), \( w^\alpha = w_1^{\alpha_1}w_2^{\alpha_2}\cdots w_n^{\alpha_n} \) and \( C_\alpha(z) \in \mathcal{O}(D), \forall |\alpha| \leq m \).

**Theorem 2.** Suppose that \( f(z, w), z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_k), n \geq 1, k \geq 1, \) is a holomorphic function in a domain \( \Omega = D_z \times \mathbb{C}_w \subset \mathbb{C}^{n+k} \), where in \( D \subset \mathbb{C}^n \) is solvable any second Cousin problem. If the set of points \( z_0 \in D \), for which the set

\[
Z_{z_0} = \{ w \in \mathbb{C}^k : f(z_0, w) = 0 \text{ is algebraic} \}
\]

is not pluripolar in \( D \), i.e., if the set

\[
\mathfrak{S}_f = \{ z_0 \in D : Z_{z_0} \text{ is algebraic in } \mathbb{C}^k \}
\]

is not pluripolar, then the function \( f(z, w) \) is represented as

\[
f(z, w) = Q_m(z, w)\varphi(z, w),
\]

where \( Q_m(z, w) \) is pseudopolynomial of some degree \( m \geq 0 \) and function \( \varphi(z, w) \in \mathcal{O}(\Omega), \varphi(z, w) \neq 0, \forall (z, w) \in \Omega \).
2. The proofs of these Theorems are based on the following two lemmas, which are also of independent interest.

**Lemma 1.** Let $D \subset \mathbb{C}^n$ be a domain, in which any second Cousin problem is solvable and $f(z, w) \in \mathcal{O}(D \times \mathbb{C}^k)$. Then there is a representation $f(z, w) = c(z)\varphi(z, w)$, where $c(z) \in \mathcal{O}(D)$, $\varphi(z, w) \in \mathcal{O}(D \times \mathbb{C}^m)$ and the dimension of the analytic set $G_{\varphi} = \{z_0 \in D : \varphi(z_0, w) \equiv 0\}$ does not exceed $n - 2$, $\dim G_{\varphi} \leq n - 2$ $(G_{\varphi} = \emptyset$ for $n = 1)$.

**Remark.** It is also easy to prove the local version of the lemma. If $f(z, w) \in \mathcal{O}(U \times V)$, where $U = U(0, \varepsilon) \subset \mathbb{C}^n$, $V = \{|w| < \varepsilon\}$ is polycylinder centred in the origin $(0, 0)$, and $f(0, 0) = 0$, then $f(z, w)$ is represented as $f(z, w) = c(z)\varphi(z, w)$, where $c(z) \in \mathcal{O}(U)$, $\varphi(z, w) \in \mathcal{O}(U \times V)$ and the dimension of the analytic set $G_{\varphi} = \{z_0 \in U : \varphi(z_0, w) \equiv 0\}$ does not exceed $n - 2$. Let $\varphi(0, w) \equiv 0$. By decomposing $\varphi(z, w)$ as above into multiple series we obtain

$$\varphi(z, w) = \sum_{k=0}^{\infty} c_k(z)w^k, \quad a_k(z) \in \mathcal{O}(U).$$

Since the space $\mathcal{O}_0$ of germs of holomorphic functions at $0 \in \mathbb{C}^n$ is Noether ring, i.e. arbitrary ideal of $\mathcal{O}_0$ has a finite basis, then the ideal $I_F$ generated by the family $F = \{c_0, c_1, \ldots\}$ has finite basis: there exits a finite system $\{c_{j_1}, \ldots, c_{j_m}\} \subset F$ such that any function $\phi \in I_F$ in some neighborhood $U' \subset U$ represented as

$$\varphi(z) = c_{j_1}(z)g_{j_1}(z) + \ldots + c_{j_m}(z)g_{j_m}(z),$$

where $g_k(z) \in \mathcal{O}(U')$, $k = 1, \ldots, m$. It follows, that in some neighborhood $U' \times V' \subset U \times V$ the function $\varphi(z, w)$ represented as

$$\psi(z, w) = c_{j_1}(z)g_{j_1}(z, w) + \ldots + c_{j_m}(z)g_{j_m}(z, w), \quad (7)$$

where $g_k(z, w) \in \mathcal{O}(U' \times V')$, $k = 1, \ldots, m$.

In applications to estimats of oscillatory integrals is very important the positive solution of the following problem

**Problem 4.** Prove justice $(7)$ with functions $g_1(z, w), \ldots, g_m(z, w) : g_j(z^0, w) \neq 0 \forall z^0 \in U', j = 1, 2, \ldots, m$.

**Lemma 2.** Let $f(z, w)$ be holomorphic in a domain $\Omega = D_z \times \mathbb{C}_w \subset \mathbb{C}^{n+1}$ and the set

$$\mathcal{G}_f = \{z \in D : n_f(z) < \infty\}$$

is not pluripolar in $D$. Then for some natural number $m \in \mathbb{N}$ there takes place $n_f(z) \leq m$, $\forall z \in D$, and $\{z \in D : n_f(z) \leq m - 1\}$ is a closed, pluripolar set.

In fact, it suffices to prove the lemma for a fixed ball $B \subset D$ such that the set $\{z \in B : n_f(z) < \infty\}$ is not pluripolar. According to Lemma 1, in the domain $B \times \mathbb{C}$ there is a representation $f(z, w) = c(z)\varphi(z, w)$, where $c(z) \in \mathcal{O}(B)$, $\varphi(z, w) \in \mathcal{O}(B \times \mathbb{C})$ and the dimension of the analytic set

$$G_{\varphi} = \{z_0 \in B : \varphi(z_0, w) \equiv 0\}$$
does not exceed \(n - 2\), \(\dim G_\varphi \leq n - 2\). Moreover, if \(\varphi(z,0) \equiv 0\), then it is represented in the form \(\varphi(z,w) = w^j \psi(z,w), \psi(z,0) \neq 0\). Thus, we can preliminarily represent the function \(f(z,w)\) in the form
\[
f(z,w) = w^j c(z) \psi(z,w),
\]
where \(j \geq 0, c(z) \in \mathcal{O}(B), \psi(z,w) \in \mathcal{O}(B \times \mathbb{C})\), \(\dim G_\varphi \leq n - 2\).

Then the function
\[
u_m(z) = \begin{cases} \ln \frac{\left|\psi(z,0)\right|}{\prod_{j=1}^{m} \alpha_j(z)} & \text{if } n_\psi(z) > m, \\ -\infty & \text{if } n_\psi(z) \leq m. \end{cases}
\]
is a plurisubharmonic function in \(B\) for any \(m \geq 0\) (see [9], Sadullaev A., Criteria algebraicity of analytic sets, Functional analysis and its application, V.6: 1, 1972, 85-86. (in Russian)). Here, \(\alpha_j(z)\) are zeros of the function \(\psi(z,w)\) by \(w\) in fixed \(z \in B\) taking into account their multiplicities and in the order not decreasing of their modules.

Note that
\[
\Im_\psi = \{z \in B : n_\psi(z) < \infty\} = \bigcup_{m=-1}^{\infty} \{z \in B : n_\psi(z) \leq m\}.
\]

It is easy to see that the sets
\[
\{z \in B : n_\psi(z) \leq m\}, m = -1,0,1,\ldots,
\]
are closed. By the condition of the lemma, for some \(m \in \mathbb{N}\) the set \(\{z \in B : n_\psi(z) \leq m\}\) is not pluripolar, but the set \(\{z \in B : n_\psi(z) \leq m - 1\}\) is pluripolar. Hence, the plurisubharmonic function \(\nu_m(z) \equiv -\infty\), which proves the validity of Lemma 2 for the ball \(B \subset D\) and, therefore, for the domain \(D\).

3. Sketch of the proof of Theorem 1. According to Lemma 2.1 \(f(z,w) = c(z) \varphi(z,w)\), where \(\dim G_\varphi \leq n - 2\). According to Lemma 2 \(D_0 = D \setminus \{z \in D : n_\psi(z) \leq m - 1\}\) is a domain, where \(n_\varphi(z) \equiv m\). In \(D_0 \times \mathbb{C}\) the function \(\varphi\) is represented as
\[
\varphi(z,w) = \psi(z,w) \prod_{j=1}^{m} (w - \alpha_j(z)) = \psi(z,w)[w^m + a_{m-1}(z)w^{m-1} + \cdots + a_0(z)],
\]
where \(a_j(z) \in \mathcal{O}(D_0), j = 0,1,\ldots,m - 1\), and \(\psi(z,w) \in \mathcal{O}(D_0 \times \mathbb{C}), \psi(z,w) \neq 0, (z,w) \in D_0 \times \mathbb{C}\). It is easy to prove that the function \(\psi(z,w)\) is locally bounded in the domain \((D \setminus G_\varphi) \times \mathbb{C}\). Consequently, it is holomorphic in it. Since, \(\dim G_\varphi \leq n - 2\), then
\[
\psi \in \mathcal{O}(D \times \mathbb{C}).
\]
Again, using Lemma 1 we decompose a function \(\psi\) into a product \(\psi(z,w) = c(z) h(z,w)\), where
\[
c(z) \in \mathcal{O}(D), h(z,w) \in \mathcal{O}(D \times \mathbb{C})
\]
and the dimension of the analytic set \( G_h = \{ z \in D : n_h(z) = -1 \} \) does not exceed \( n - 2 \). Since, besides \( \psi(z, w) \neq 0 \), in \( D_0 \times \mathbb{C} \), then \( h(z, w) \neq 0 \) in \( D_0 \times \mathbb{C} \), and since \( \dim G_h \leq n - 2 \), then \( h(z, w) \neq 0 \) in \( D \times \mathbb{C} \). From here

\[
f(z, w) = c(z)\varphi(z, w) = c(z)\psi(z, w)[w^m + a_{m-1}(z)w^{m-1} + \cdots + a_0(z)] = c(z)\tilde{c}(z)h(z)[w^m + a_{m-1}(z)w^{m-1} + \cdots + a_0(z)] = [c_m(z)w^m + c_{m-1}(z)w^{m-1} + \cdots + c_0(z)]h(z, w),
\]

where \( c_k(z) \in \mathcal{O}(D), \ k = 0, 1, \ldots, m, h(z, w) \in \mathcal{O}(D \times \mathbb{C}), h(z, w) \neq 0 \ \forall (z, w) \in D \times \mathbb{C}. \)

4. Proof of Theorem 2. According to Lemma 1, without loss of generality, we can assume that the dimension of an analytic set \( \mathcal{G}_f = \{ z_0 \in D : f(z_0, w) \equiv 0 \} \) does not exceed \( n - 2 \), \( \dim \mathcal{G}_f \leq n - 2 \). Since the set \( \mathcal{G}_f = \{ z \in D : Z_z \text{ is algebraic in } \mathbb{C}^k \} \) is not pluripolar in \( D \) and \( \mathcal{G}_f = \bigcup_{j=0}^{\infty} \mathcal{G}_f^j \), where \( \mathcal{G}_f^j = \{ z \in \mathcal{G}_f : \deg Z_z \leq j \} \), then there exists \( m \geq 0 \) such that \( \mathcal{G}_f^j \) is not pluripolar, but \( \mathcal{G}_f^{j-1} \) is closed pluripolar set in \( D \).

In the space \( \mathbb{C}^k \) we fix a complex line

\[
l = l_u : w = u\xi, \ u = (u_1, u_2, \ldots, u_k) \in \mathbb{C}^k,
\]

\( \xi \in \mathbb{C} \) — parameter, and consider the restriction \( f^u(z, \xi) = f(z, u\xi) \in \mathcal{O}(D_z \times C\xi) \).

For arbitrary fixed \( u \in \mathbb{C}^k \) and \( z \in \mathcal{G}_f^m \) the entire function \( f(z, u\xi) \) of variables \( \xi \in \mathbb{C} \) has finite \((\leq m)\) number of zeros. Since \( \mathcal{G}_f^m \) is not pluripolar, then according to Lemma 2 \( \mathcal{G}_f^m = D \ \forall u \in \mathbb{C}^k \). It means, that for arbitrary fixed \( z \in D \) and \( u \in \mathbb{C}^k \) the entire function \( f(z, u\xi) \) of variable \( \xi \in \mathbb{C} \) has finite \((\leq m)\) number of zeros. Then as it was proven in [9] for arbitrary fixed \( z \in D \) the set \( Z_z \) is algebraic of \( \deg Z_z \leq m \), i.e. \( \mathcal{G}_f^m = D, \ \mathcal{G}_f^{m-1} = D \).

It follows from Theorem 1 that the restrictions \( f(z, u\xi), u \in \mathbb{C}^k \) are represented as

\[
f(z, u\xi) = [c_m(z, u)\xi^m + \cdots + c_0(z, u)]\varphi(z, \xi, u), \quad (8)
\]

where

\[
c_k(z, u) \in \mathcal{O}(D \times \mathbb{C}^k), \ k = 0, 1, \ldots, m,
\]

\( \varphi(z, \xi, u) \in \mathcal{O}(D \times \mathbb{C} \times \mathbb{C}^k), \varphi(z, \xi, u) \neq 0, \forall (z, \xi, u) \in D \times \mathbb{C} \).

Since the left side of (8) does not change in the transformation \( u \to \lambda u, \xi \to \frac{\xi}{\lambda}, \) the pseudopolynomial \( c_m(z, u)\xi^m + c_{m-1}(z, u)\xi^{m-1} + \cdots + c_0(z, u) \) in this transformation also does not change:

\[
c_m(z, \lambda u)\xi^m + c_{m-1}(z, \lambda u)\xi^{m-1} + \cdots + c_0(z, \lambda u) =
\]
\[ f(z, u) = c_m(z, u)\xi^m + c_{m-1}(z, u)\xi^{m-1} + \cdots + c_0(z, u). \]

Consequently,
\[ c_k(z, \lambda u) = \lambda^k c_k(z, u), \quad k = 0, 1, \ldots, m, \]
and functions \( c_k(z, u), \quad k = 0, 1, \ldots, m, \) polynomials with respect to the variable \( u \in \mathbb{C}^k, \) \( c_k(z, \lambda u) = \lambda^k c_k(z, u). \) Hence,
\[
\begin{align*}
&c_m(z, u)\xi^m + c_{m-1}(z, u)\xi^{m-1} + \cdots + c_0(z, u) = \\
&= c_m(z, \xi u) + c_{m-1}(z, \xi u) + \cdots + c_0(z, \xi u) = \\
&= c_m(z, w) + c_{m-1}(z, w) + \cdots + c_0(z, w) = Q_m(z, w)
\end{align*}
\]
is a pseudopolynomial of degree \( m \) in the domain \( \Omega = D_z \times \mathbb{C}_u^k. \) The ratio
\[
\frac{f(z, w)}{Q_m(z, w)} = \varphi(z, w)
\]
is holomorphic in \( \Omega = D_z \times \mathbb{C}_u^k, \) \( \varphi(z, w) \in \mathcal{O}(\Omega) \) and \( \varphi(z, w) \neq 0. \) The theorem is proved.

5. Analogy of the Weierstrass theorem holds true for the real analytic functions: let \( f(x, t) \) to be real-analytic and real-valued function at the point \( (0, 0) \in \mathbb{R}^n_x \times \mathbb{R}^l_t, \) such that \( f(0, 0) = 0, \) but \( f(0, t) \not\equiv 0. \) The corresponding holomorphic function \( f(z, w) \) by Weierstrass preparation theorem in some neighborhood \( U = V \times W \) of \((0, 0)\) is represented as
\[
f(z, w) = [w^m + c_{m-1}(z)w^{m-1} + \cdots + c_0(z)]\varphi(z, w),
\]
where \( m \geq 1 \) is the order of zero of \( f(0, t) \) at \( t = 0, c_k(z), \quad k = 0, 1, \ldots, m - 1, \) are holomorphic in \( V, \) \( c_k(z_0) = 0 \) and \( \varphi(z, w) \) is holomorphic and \( \not\equiv 0 \) in \( U. \) Put \( f(z, w) = \sum_{j=0}^{\infty} a_j(z)w^j, \quad a_j(z) \in O(V). \) Since \( f(x, t) \) is real-valued, then all Taylor coefficients \( a_j(x) \) are real-valued in \( V \cap \mathbb{R}^n. \) It follows, that if \( w = \alpha \) is zero of the function \( f(x, w), \) i.e. \( f(x, \alpha) = 0, \) then the conjugate \( \overline{\alpha} \) also is zero, \( f(x, \overline{\alpha}) = 0. \)

Therefore, the Weierstrass polynomial \( t^m + c_{m-1}(x)t^{m-1} + \cdots + c_0(t) \) is real-valued, \( c_k(x), \quad k = 0, 1, \ldots, m - 1, \) are real-valued functions in \( V \cap \mathbb{R}^n. \) From \( f(x, t) = [t^m + c_{m-1}(x)t^{m-1} + \cdots + c_0(x)]\varphi(x, t) \) we have, that \( \varphi(x, t) \not\equiv 0 \) and is real-analytic, real-valued function in \( U \cap \mathbb{R}^{n+1}. \) Consequently,
\[
f(x, t) = [t^m + c_{m-1}(x)t^{m-1} + \cdots + c_0(x)]\varphi(x, t),
\]
where \( c_k(x), \quad k = 0, 1, \ldots, m - 1, \) \( \varphi(x, t) \) are real-analytic, real-valued functions, \( \varphi(x, t) \not\equiv 0. \)

It is true also a global real analogy without the condition \( f(0, t) \not\equiv 0. \) Let \( f(x, t) \not\equiv 0 \) to be real-analytic and real-valued function in a domain \( \Omega = D_x \times \mathbb{R}_t \subset \mathbb{R}^{n+1}. \)

For arbitrary fixed \( x_0 \in D_x \) the function \( f(x_0, t) \) continues holomorphically to \( \mathbb{C} \) as an entire function. On the other hand, the function \( f(x, t) \) as real-analytic in the domain \( \Omega = D_x \times \mathbb{R}_t \subset \mathbb{R}^{n+1} \) extends holomorphically to some neighborhood \( \hat{\Omega} \subset \mathbb{C}_x^n \times \mathbb{C}_w, \) \( \hat{\Omega} \supset \Omega. \) We denote the extended to \( \hat{\Omega} \) function as \( \hat{f}, \) \( \hat{f}|_{\hat{\Omega}} = f. \) Since
\[ \hat{\Omega} \cap \{ z = x \} = \mathbb{C} \] for arbitrary fixed \( x \in D \), and \( D \subset \mathbb{R}_+^n \subset \mathbb{C}_z^n \), then there exists a domain \( \hat{D}_z \supset D_x \), such that \( \hat{f} \) is holomorphic in \( \hat{D} \times C_w \). Without loss of generality, we can assume that any second Cousin problem is solvable in \( \hat{D} \). Let us denote by \( n_f(x_0) \) the number of zeros of the function \( \hat{f}(x_0, w) \) of variable \( w \in \mathbb{C} \) taking into account multiplicity. We put \( n_f(x_0) = -1 \) if \( \hat{f}(x_0, w) \equiv 0 \).

**Theorem 3.** Let \( f(x, t) \) be real analytic and real valid function in a domain \( \Omega = D_x \times \mathbb{R}_t^k \subset \mathbb{R}^{n+k} \). If the set \( \Im(f) := \{ x \in D : n_f(x) < \infty \} \) is not pluripolar in \( \mathbb{C}^n \), \( D \subset \mathbb{R}^n \subset \mathbb{C}_z^n \), then the function \( f(x, t) \) is represented as,

\[
 f(x, t) = [c_m(x)t_n + c_m-1(x)tm-1 + \cdots + c_0(x)] \varphi(x, t),
\]

where \( c_k(x), k = 0, 1, \ldots, m, \) are real analytic in \( D \) and \( \varphi(x, t) \) is real analytic in \( \Omega, \varphi(x, t) \neq 0 \) for any \( (x, t) \in \Omega \).

There is also a multidimensional (in \( t \)) analogue of this theorem in the following form. The pseudopolynomial of degree \( m \) in a domain \( \Omega = D_x \times \mathbb{R}_t^k \subset \mathbb{R}^{n+k} \) is an expression in the form:

\[
 Q_m(x, t) = \sum_{|\alpha|=m} C_\alpha(x) t^\alpha + \sum_{|\alpha|=m-1} C_\alpha(z) t^\alpha + \cdots + C_0(x),
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n, w^\alpha = w_1^{\alpha_1}w_2^{\alpha_2}\ldots w_n^{\alpha_n} \) and \( C_\alpha(x) \) is real analytic function in \( D \) \( \forall |\alpha| = m \).

**Theorem 4.** Suppose that

\[
 f(x, t), x = (x_1, \ldots, x_n), t = (t_1, \ldots, t_k), n \geq 1, k \geq 1,
\]
is a real analytic function in a domain \( \Omega = D_x \times \mathbb{R}_t^k \subset \mathbb{R}^{n+k} \). If the set of points \( x_0 \in D \), for which the set \( Z_{x_0} = \{ w \in \mathbb{C}^k : \hat{f}(x_0, w) = 0 \) is algebraic \} is not pluripolar in \( \mathbb{C}^n \), \( D \subset \mathbb{R}^n \subset \mathbb{C}_z^n \), i.e., if the set \( \Im(f) = \{ x_0 \in D : Z_{x_0} \text{ is algebraic in } \mathbb{C}^k \} \) is not pluripolar, then the function \( f(x, t) \) is represented as

\[
 f(x, t) = Q_m(x, t) \varphi(x, t),
\]

where \( Q_m(x, t) \) is a real valid pseudopolynomial of some degree \( m \geq 0 \) and function \( \varphi(x, t) \) is real analytic real valid in \( \Omega, \varphi(x, t) \neq 0 \) for any \( (x, t) \in \Omega \). Here the function \( \hat{f}(x, t) \) is holomorphic extension of \( f(x, t) \) to \( \hat{\Omega} \supset \Omega, \hat{f}|_{\hat{\Omega}} = f \).

Let now \( f(x, t) \neq 0 \) to be real-analytic and real-valued function in a neighborhood of the point \( (0, 0) \in \mathbb{R}^{n+1} \), such that \( f(0, 0) = 0 \).

**Theorem 5.** There exist a real analytic manifold \( Y \) and a map \( \pi : Y \rightarrow \mathbb{R}^n \) which is a finite composition of blowing-up such that for any point \( y^0 \in Y \) there exists a chart \( (y_1, \ldots, y_n) \) with the center at the point \( y^0 \) in a neighborhood of that point the following relation:

\[
 f(\pi(y), t) = (y_1 - y_0^{0k_1}) \cdots (y_n - y_0^{0k_n}) p(y, t) g(y, t),
\]

holds true. Here \( k_j \in \mathbb{N}, j = 1, 2, \ldots, n, g(y, t) \) is a real analytic function, \( g(y_0, 0) \neq 0 \), and \( p(y, t) \) is a unitary pseudopolynomial.

\[
p(y, t) = t^m + d_1(y)t^{m-1} + \cdots + d_m(y),
\]

\( d_1, \ldots, d_m \) are real analytic functions at the point \( y_0 \) and \( d_l(y_0) = 0, l = 1, 2, \ldots, m \).

Theorem 5 had been proved in the paper of I. Ikramov [3].
2 Some related results

Here we give only one result, where Weierstrass-type theorems are used to estimate the oscillatory integrals. Suppose $S(\sigma) \subset \mathbb{R}^{n+1}$ be a family of smooth (analytic) hyper-surfaces smoothly (analytically) depending on the parameters $\sigma \in U \subset \mathbb{R}^m$, where $U$ is a neighborhood of a fixed point $\sigma^0 \in \mathbb{R}^m$ say of the point $\sigma^0 = 0$.

Denote by $\Lambda_l(x)$, $(0 \leq l \leq n)$ the class of smooth hyper-surfaces having at least $l$ non-vanishing principal curvatures at the point $x$. Further, the relation $S(\sigma) \subset \Lambda_l$ means that at every point $x \in S(\sigma)$ we have $S(\sigma) \subset \Lambda_l(x)$. We have classical Littman W. theorem (see [5]), that if a hyper-surface $S(\sigma)$ belongs to the class $\Lambda_l$ then the integral (2) has the following estimate:

$$\hat{d}\mu_\sigma(\xi) = O(|\xi|^{-l/2}) \quad (\text{as } |\xi| \to +\infty).$$

Moreover, the last asymptotic relation holds true locally uniformly with respect to the parameters $\sigma$ and directions of the vector $\xi$. In the sense that for any compact set $\Delta \subset U$ there exists a constant $C(\Delta)$ such that the estimate

$$|\hat{d}\mu_\sigma(\xi)| \leq \frac{C(\Delta)}{|\xi|^{l/2}}$$

holds true for any $\sigma \in \Delta$ and $\xi \in \mathbb{R}^{n+1}$.

**Theorem 6.** Suppose $S(\sigma)$, $\sigma \in \mathbb{R}^m$, is an analytic family of hyper-surfaces satisfying the conditions:

1) $S(0) \subset \mathbb{R}^{n+1}$ is an analytic hyper-surface and $S(0) \in \Lambda_{n-1}$;
2) Denote by $K(x, \sigma)$ the Gaussian curvature at the point $x \in S(\sigma)$. Assume $K(x, 0) \not\equiv 0$ then the following statements are hold:

(i) There exist a neighborhood $V \times U \subset \mathbb{R}^{n+1} \times \mathbb{R}^m$ and $p_m > 2$ such that for any $\psi \in C^\infty_0(V)$ the following inclusion $M_\sigma \in L^{p_m}(\Sigma^n)$ holds, where $M_\sigma$ is the corresponding to $\hat{d}\mu_\sigma(r\omega)$ Randol maximal function. Moreover, the following integral

$$\int_{\Sigma^n} M_\sigma^{p_m} d\omega$$

is uniformly bounded with respect to $\sigma \in U \subset \mathbb{R}^m$.

(ii) If $p_m > 2(n+1)/n$ then $\hat{d}\mu_\sigma \in L^p(\mathbb{R}^{n+1})$ for any $p > 2(n+1)/n$. Moreover, the following integral

$$\int_{\mathbb{R}^{n+1}} |\hat{d}\mu_\sigma(\xi)|^p d\xi$$

is uniformly bounded with respect to $\sigma \in U$ for any fixed $p > 2(n+1)/n$.

(iii) If $2 < p_m \leq 2(n+1)/n$ then $\hat{d}\mu_\sigma \in L^p(\mathbb{R}^{n+1})$ for any $p > (2n+2-p_m)/(n-1)$. Moreover, the following integral

$$\int_{\mathbb{R}^{n+1}} |\hat{d}\mu_\sigma(\xi)|^p d\xi$$

is uniformly bounded with respect to $\sigma \in U$ for any fixed $p > (2n+2-p_m)/(n-1)$.
Remark. If $S$ is the cylinder with the spherical base then we have $S \in \Lambda_{n-1}$ and $K \equiv 0$. In this case $M \not\in L^2(\Sigma^n)$. Moreover, if $n = 2$ then $\hat{d}\mu \not\in L^4(\mathbb{R}^3)$ for appropriate amplitude function and $\hat{d}\mu \in L^p(\mathbb{R}^3)$ for any $p > 4$ and $\psi \in C^\infty_0(\mathbb{R}^3)$.

The following results shows sharpness of our results.

Proposition. For any positive number $\varepsilon$ there exists a hyper-surface $S \in K_1$ in $\mathbb{R}^3$ with $K(x) \not\equiv 0$ such that $\hat{d}\mu \not\in L^{(4-\varepsilon)}(\mathbb{R}^3)$.

Proof of the Theorem 6. Suppose $S(\sigma)$ is a family of real-analytic hyper-surfaces. By using stationary phase arguments we reduce our integral to one dimensional integral satisfying the conditions of the Lemma 3 (see below) on estimates for one-dimensional oscillatory integrals. For the one dimensional integral we use estimate which follows from the classical Weierstrass Theorem.

Note that, the Fourier transform $\hat{d}\mu$ locally can be written as an oscillatory integral. Indeed $S(\sigma)$ after possible Euclidian motion locally can be written as the graph of an analytic function, so without loss of generality we may assume

$$S(\sigma) = \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1}: x_{n+1} = \phi(x', \sigma)\},$$

where $\phi(x', \sigma)$ is a real-analytic function satisfying the conditions: $\phi(0, 0) = 0$, $\nabla \phi(0, 0) = 0$. Then the surface integral

$$\hat{d}\mu_\sigma(\xi) = \int_{S(\sigma)} e^{i(x, \xi)} d\mu(x)$$

can be written as the following multiple integral:

$$\int_{\mathbb{R}^n} e^{i(x', \xi) + \xi_{n+1}\phi(x', \sigma')} d\mu(x', \phi(x', \sigma')).$$

Passing to the project coordinates $\lambda = \xi_{n+1}, s_j = \frac{\xi_j}{\lambda}$ $(j = 1, \ldots, n)$ we get an oscillatory integral:

$$J(\lambda, s, \sigma) := \int_{\mathbb{R}^n} e^{i\Phi(x', s, \sigma)} a(x') dx',$$

where

$$\Phi(x', s, \sigma) = \phi(x', \sigma) + (x', s).$$

Here the function $\phi(x', \sigma)$ is real analytic function satisfying the conditions: $\nabla \phi(0, 0) = 0$, rank Hess $\phi(0, 0) = \text{rank det}\{\partial_j \partial_k \phi(0, 0)\}_{j, k=1}^n \geq n - 1$, which is equivalent to the condition $S(0) \in \Lambda_{n-1}$.

If $|s| > \varepsilon$ (where $\varepsilon > 0$ is a fixed positive real number) then by using integration by parts arguments we obtain

$$J(\lambda, s, \sigma) = O(|s\lambda|^{-N}) \quad \text{as} \quad |\lambda| \to +\infty,$$

for any $N$ provided $a$ is a smooth function concentrated in a sufficiently small neighborhood of the origin. Thus, we may assume that $|s| \leq \varepsilon$.

Then without loss of generality assuming

$$\text{det}\{\partial_j \partial_k \phi(0, 0)\}_{j, k=1}^{n-1} \neq 0$$

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we can use classical Morse Lemma: there exists a diffeomorphic mapping \( \varphi(x', s, \sigma) \) of \((0, 0, 0) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \) such that the phase function \( \Phi(x, s, \sigma) \) can be written in the form:

\[
\Phi(\varphi(y, s, \sigma), s, \sigma) := \Phi_1(y_1, s_2, \ldots, s_n, \sigma) + s_1 y_1 + Q(y_2, \ldots, y_n), \quad y = (y_1, y_2, \ldots, y_n),
\]

where \( \Phi_1(y_1, s_2, \ldots, s_n, \sigma) \) is a real analytic function, \( Q := (Ay'', y'') \) is a non-degenerate quadratic form of variables \((y_2, \ldots, y_n)\) given by invertible symmetric matrix \( A \). Moreover, the condition

\[
\frac{\partial^2 \Phi_1(y_1, s_2, \ldots, s_n, 0)}{\partial y_1^2} \neq 0
\]

is equivalent to the condition \( K(x, 0) \neq 0 \).

Thus the oscillatory integral \( J(\lambda, s, \sigma) \) after change of variables can be written as the following iterated integral

\[
J(\lambda, s, \sigma) = \int_{\mathbb{R}} e^{i\lambda(\varphi(y_1, s_2, \ldots, s_n, \sigma) + s_1 y_1)} \times \left( \int_{\mathbb{R}^{n-1}} e^{i\lambda Q(y_2, \ldots, y_n)} a_1(y, s) dy_2 \ldots dy_n \right) dy_1,
\]

with smooth amplitude function \( a_1 \in C_0^\infty(U \times V) \).

Then by classical stationary phase arguments we have

\[
\int_{\mathbb{R}^{n-1}} e^{i\lambda Q(y_2, \ldots, y_n)} a_1(y, s) dy_2 \ldots dy_n =
\]

\[
= c a_1(y_1, 0, \ldots, 0, s) \lambda^{-\frac{n-1}{2}} + O(|\lambda|^{-\frac{n+1}{2}}) \text{ as } |\lambda| \to +\infty,
\]

where

\[
c = \frac{e^{i\text{sign}(Q)\frac{\pi}{4}}}{\sqrt{\pi^{n-1} |\det A|}}.
\]

Thus, the oscillatory integral \( J(\lambda, s, \sigma) \) can be written as

\[
J(\lambda, s, \sigma) =
\]

\[
= c \lambda^{\frac{1-n}{2}} \int_{\mathbb{R}} e^{i\lambda(\Phi_1(y_1, s_2, \ldots, s_n, \sigma) + s_1 y_1)} a_1(y_1, 0, \ldots, 0, s) dy_1 + O(|\lambda|^{-\frac{n+1}{2}}) \text{ as } |\lambda| \to +\infty.
\]

So, the problem on investigation behavior of the the oscillatory integral \( J(\lambda, s, \sigma) \) can be reduced to the estimation problem of the one-dimensional oscillatory integral

\[
J_1(\lambda, s, \sigma) = \int_{\mathbb{R}} e^{i\lambda(\Phi_1(y_1, s_2, \ldots, s_n, \sigma) + s_1 y_1)} a_1(y_1, 0, \ldots, 0, s) dy_1.
\]
The phase function $\Phi_1(y_1, s', \sigma) + s_1 y_1$, where $s' = (s_2, \ldots, s_n)$, can be written in the form:

$$\Phi_1(y_1, s', \sigma) + s_1 y_1 = \Phi_2(y_1, s', \sigma) + \tilde{s}_1 y_1 + \sigma_0(s', \sigma),$$

where

$$\tilde{s}_1 = s_1 + \frac{\partial \Phi_1(0, s', \sigma)}{\partial y_1}, \quad \sigma_0(s', \sigma) = \Phi_1(0, s', \sigma)$$

and

$$\Phi_2(y_1, s', \sigma) = \Phi_1(y_1, s', \sigma) - \frac{\partial \Phi_1(0, s', \sigma)}{\partial y_1} y_1 - \sigma_0(s', \sigma).$$

Then the function $\Phi_2(y_1, s', \sigma)$ can be written as

$$\Phi_2(y_1, s', \sigma) = y_1^2 G(y_1, s', \sigma),$$

where $G$ is a real analytic function satisfying the condition $G(y_1, s', 0) \neq 0$.

We denote $\tilde{s}_1$ again by $s_1$ and $y_1$ by $x$. Then we define the phase function

$$F(x, s, \sigma) = x^2 G(x, s', \sigma) + s_1 x,$$

and consider the corresponding oscillatory integral:

$$J(\lambda, s, \sigma) := \int_{\mathbb{R}} e^{i\lambda F(x, s, \sigma)} b(x, s, \sigma) dx, x \in \mathbb{R}.$$
Consequently, there exists a natural number $N$ such that ideal $I$ generated by elements $\{c_k\}_{k=0}^N$. Also naturally can be defined the ideal $I_0 := \langle \{c_k(\cdot, 0)\}_{k=0}^\infty \rangle$, which is generated by elements $\{c_k(\cdot, 0)\}_{k=0}^N$. In particular, for the function

$$c^2(s', \sigma) := \sum_{l=0}^{N} c_l^2(s', \sigma)$$

we have relation $c^2(s', 0) \equiv 0$. Consequently there exits a positive number $\delta > 0$ and a neighborhood of the origin $V' \times W \subset \mathbb{R}^{n-1} \times \mathbb{R}^m$ such that for a point $\sigma \in U$ the following integral

$$\int_{V'} |c(s', \sigma)|^\delta$$

uniformly bounded with respect to $\sigma \in U$.

To estimate the oscillatory integral $J(\lambda, s, \sigma)$ first consider the case $c(s', \sigma) < |s_1|$. Then the function $F(x, s, \sigma)/(x^2 c(s', \sigma))$ and its derivatives are bounded. Thus we can use integration by parts arguments and have:

$$|J(\lambda, s, \sigma)\chi_{\{c(s', \sigma) < |s_1|\}}(s, \sigma)| \leq \frac{C}{|\lambda s_1|^{1/2}};$$

with a constant $C = C(s, \sigma)$, where $\chi_{\{c(s', \sigma) < |s_1|\}}$ is the indicator function of the set $\{c(s', \sigma) < |s_1|\}$.

Consider the case $c(s', \sigma) > |s_1|$. We put

$$F_1(x, s, \sigma) := \frac{F(x, s, \sigma)}{c(s', \sigma)} \quad \text{and} \quad \varsigma_1 := \frac{s_1}{c(s', \sigma)}.$$

The the function $F_1(x, s, \sigma)$ is a smooth (analytic). We fix $\varsigma_1 = \varsigma_1^0 \in [-1, 1]$ and consider the function

$$\Phi^0(x, y, \varsigma_1) := \Phi_1(x, \pi(y)) + \varsigma_0^0 x + (\varsigma_1 - \varsigma_0^0) x,$$

where $y$ is a local chart on a real analytic manifold $Y$ defined by Theorem 5, and a map $\pi: Y \rightarrow \mathbb{R}^n$ is a finite composition of blowing-up.

Now, since $k_j$ in Theorem 5 are finite, there exists a function $\Psi(y, \varsigma_1)$ such that for the oscillatory integral the following estimate:

$$|J(\lambda, s, \sigma)|\chi_{\{|c(s', \sigma)| \geq |s_1|\}} \leq \frac{C(s, \sigma)\Psi(\sigma, \frac{s_1}{c(s', \sigma)})}{|c(s', \sigma)\lambda|^{1/2}};$$

holds. Moreover, the following integral:

$$\int_{-1}^{1} (\Psi(y, \varsigma_1))^p d\varsigma_1$$

is uniformly bounded on a compact subset of the manifold $Y$ due to finite covering property of the map $\pi$. Consequently,

$$|J(\lambda, s, \sigma)| \leq C(s, \sigma) \left\{ \frac{1}{|\lambda s_1|^{1/2}} + \frac{\Psi(\sigma, \frac{s_1}{c(s', \sigma)})}{|c(s', \sigma)\lambda|^{1/2}} \right\}.$$

\[\square\]
References


